Math 374
Test 2

Name:_____

Directions: Complete each the following problems. Show and explain your work for full credit. Correct answers without reasoning *will not receive full credit!*

(1) The *Catalan* numbers are defined by the following recurrence relation:

$$C(0) = 1$$
$$C(1) = 1$$
$$C(n) = \sum_{k=1}^{n} C(k-1)C(n-k) \quad \text{for } n \geq 2.$$

Calculate $C(2), C(3)$, and $C(4)$.

$C(2) = \sum_{1}^{2} C(k-1)C(2-k)$

$= C(0)C(1) + C(1)C(0)$

$1 + 1 = 2$

$C(3) = \sum_{1}^{3} C(k-1)C(3-k)$

$= C(0)C(2) + C(1)C(1) + C(2)C(0)$

$2 + 1 + 2 = 5$

$C(4) = C(0)C(3) + C(1)C(2) + C(2)C(1) + C(3)C(0)$

$5 + 2 + 2 + 5 = 14$

(2) A sequence is defined by

$$T(1) = 4,$$
$$T(2) = -2,$$
$$T(n) = -T(n-1) + 2T(n-2) \quad \text{for } n \geq 3.$$

Find a closed form for $T(n)$. (That is, solve the recurrence relation.)

$t^2 = -t + 2$

$t^2 + t - 2 = 0$

$(t+2)(t-1) = 0$

$t = -2, 1.$ $\quad r_1 = -2 \quad r_2 = 1$

$T(n) = p r_1^{n-1} + q r_2^{n-1}$

$4 = p + q$ $\quad 6 = 3p \rightarrow p = 2$

$-2 = -2p + q$ $\quad q = 2.$

$$T(n) = 2(-2)^{n-1} + 2.$$

(3) Let $A = \{1, 2, 3, 4\}$ and $B = \{a, \{b\}, \{a, \{b\}\}\}$.

(a) List the elements of $A \cup B$.

$A \cup B = \{1, 2, 3, 4, a, \{b\}, \{a, \{b\}\}\}$

(b) What is the cardinality of $A \times B$?

$4 \times 3 = 12$

(c) What is the cardinality of $\mathcal{P}(\mathcal{P}(B))$?

$2^{2^3} = 2^8 = 256$

(d) For each of the following, circle T if the statement is true, and F if it is false.

(T) F $\{a, \{b\}\} \subseteq B$.

(T) F $\{a, \{b\}\} \in B$.

T (F) $\{1, 2\} \in A$.

(T) F $\{1, 2\} \in \mathcal{P}(A)$.

T (F) $\{a, b\} \subseteq B$.

(T) F $\emptyset \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

(T) F $\emptyset \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

(4) License plates in a certain state come in two types: Standard plates have three letters followed by three numbers, where the first number cannot be 0. Vanity plates have two letters followed by three numbers, where the two letters must be different, and again the first number cannot be zero. How many different lisence plates are possible?

Standard: $26 \cdot 26 \cdot 26 \cdot 9 \cdot 10 \cdot 10$

Vanity: $26 \cdot 25 \cdot 9 \cdot 10 \cdot 10$.

Total $26^3 \cdot 900 + 26 \cdot 25 \cdot 900$

(5) A survey of 150 college students reveals the following about their transportation options. 83 own a car, 97 own a bike, and 28 own a skateboard. 53 own a car and a bike, 14 own a car and a skateboard, 7 own a bike and a skateboard, and 2 own all three. How many students do not own any of the three?

A = Car B = Bike C = Skateboard.

$|A \cup B \cup C| = 83 + 97 + 28 - 53 - 14 - 7 + 2$

By I - E.

$208 - 74 + 2$

$210 - 74 = 136.$

Own none is $(A \cup B \cup C)^c$.

$|(A \cup B \cup C)^c| = 150 - 136$

$= 14$

(6) I have 18 marbles. 3 are red, 6 are blue, and 9 are green.

(a) How many marbles do I have to choose to guarantee I get at least one of *every* color?

16 ; if less, could use just green + blue marbles.
(ie, Max w/ Thonly R+B = 9, Max w/ Thonly R+G = 12,
Max w/ Thonly B+G = 15, so Min w/ Th ● all 3
is **16**

(b) How many do I have to choose to guarantee I have at least two of *some* color?

4. (By Pigeonhole. Colors are holes, Marbles are Pigeons)

(7) (a) How many different ways can I rearrange the letters of the word HAWAIIAN?

H, W, 3 A's, 2 I's, N.
Place H. Then W Then A's Then I's Then N in
_ _ _ _ _ _ _ _ $8 \cdot 7 \cdot \binom{6}{3} \cdot \binom{3}{2} \cdot 1$ ways

(b) How many of these start with the letter H?

~~Skip the H, tower slots by 1.~~ only 1 choice for H.
$7 \cdot \binom{6}{3} \cdot \binom{3}{2} \cdot 1$.

(8) A certain restaurant offers three types of appetizers (chicken fingers, quesadillas, and jalapeno poppers). A party of 8 people comes to the restaurant and each orders an appetizer. How many different appetizer trays could the kitchen have to make? (*Note:* The person who ordered a particular appetizer doesn't alter the tray. The tray only counts how many people ordered each type of appetizer.)

3 appetizers are the bins, want to distribute 8 items to them.

CF Q JP

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10 slots, 2 separators. $\binom{10}{2}$ ways

(9) Find the coefficient of x^6 in $(3x - 1/2)^9$.

$$(3x - \tfrac{1}{2})^9 = \sum_{i=0}^{n} (3x)^{n-i} (-\tfrac{1}{2})^i \binom{n}{i}$$

$i = 3$ for the x^6 term.

$$(3x)^6 (-\tfrac{1}{2})^3 \binom{6}{3}$$

$$= \boxed{-3^6 \cdot (\tfrac{1}{2})^3 \binom{6}{3}} x^6$$

Coefficient.