Math 374
Test 1

Name: Key

(1) (6 pts) Verify that the following is a tautology using a truth table.

$$(A \vee B) \wedge A' \to B$$

A	B	A'	$(A \vee B)$	$(A \vee B) \wedge A'$	$(A \vee B) \wedge A' \to B$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

(2) (6 pts) Construct a proof sequence for the following argument.

$$(A' \to B') \wedge B \wedge (A \to C) \to C$$

1) $A' \to B'$ hyp

2) B hyp

3) $A \to C$ hyp

4) A MT, 1, 2

5) C MP; 3,4

(3) (4 pts) The following argument is *not* valid. Give an interpretation that demonstrates this.

$$(\forall x)\,[(\exists y)P(x,y) \wedge (\exists y)Q(x,y)] \rightarrow (\forall x)(\exists y)\,[P(x,y) \wedge Q(x,y)]$$

Let Domain be the integers, $P(x,y)$ say $y < x$,
$Q(x,y)$ say $y > x$.
LHS says "for any integer, there is one that's smaller, and one that's larger." which is true. The RHS says "for every integer there's a single number that is larger and smaller than the first one".

Th. 3.3
[illegible]

(4) (7 pts) Construct a proof sequence for the following argument (this statement is the *converse* of the previous statement).

$$(\forall x)(\exists y)\,[P(x,y) \wedge Q(x,y)] \rightarrow (\forall x)\,[(\exists y)P(x,y) \wedge (\exists y)Q(x,y)]$$

Removed

(5) (7 pts) Verify the correctness of the following program with given precondition and postcondition.

$\{y = 0\}$
if $y < 5$ **then**
$y = y + 1$
else
$y = 5$
end if
$\{y = 1\}$

<u>Need to check</u>

1)

$\{y = 0 \wedge y < 5\}$
$y = y + 1$
$\{y = \cancel{1}\}$

By assignment, we get

$\{y + 1 = 1\}$
$y = y + 1$
$\{y = 1\}$.

The top says $\{y = 0\}$ which is equivalent to $\{y = 0 \wedge y < 5\}$.

and

2)

$\{y = 0 \wedge y \geq 5\}$
$y = 5$
$\{y = 1\}$

This is true because the precondition is always false.

(6) (10 pts) Prove *ONE* of the following three statements.

(a) If x is a rational number, then $\sqrt{2}+x$ is an irrational number. (Hint: Contradiction, using the fact that $\sqrt{2}$ is irrational.)

(b) If $x \neq y$ and $a \neq 0$, then $\frac{x}{y} \neq \frac{x+a}{y+a}$. (Hint: Contrapositive.)

(c) If two integers are each divisible by an integer n, then the sum of the two integers is divisible by n.

a) Suppose That $\sqrt{2}+x$ is rational. Then we can write

$\sqrt{2} + \frac{a}{b} = \frac{c}{d}$ ~~since~~ where $a,b,c,d \in \mathbb{Z}$, since

x is also rational. Then

$$\sqrt{2} = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{db}.$$

This says That $\sqrt{2}$ is rational, which is a contradiction.

Thus $\sqrt{2}+x$ must be irrational.

b) We prove The contrapositive, That if $\frac{x}{y} = \frac{x+a}{y+a}$, Then $x=y$ or $a=0$.

Since $\frac{x}{y} = \frac{x+a}{y+a}$, we know That $x(y+a) = y(x+a)$

rearranging, This says $(x-y)a = 0$ Hence $x=y$, or $a=0$.

c) Let a,b each be divisible by n. Then we can write $a=nk$, $b=nl$ for integers l, k.

Then $a+b = n(k+l)$, which says That $a+b$ is divisible by n also.

(7) (10 pts) Prove *ONE* of the following two statements using mathematical induction.

(a) For all $n \geq 4$, we have $3^n > n^3$.

(b) For all $n \geq 1$, we have $3 \mid (4^n - 1)$.

a) Base Case $n = 4$:

Note That $3^4 = 81 > 64 = 4^3$.

Inductive Let $k \geq 4$, and assume $3^k > k^3$.

Then
$$
\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&> 3k^3 \quad \text{by induction} \\
&\geq k^3 + 8k^2 \quad (\text{since } k \geq 4) \\
&\geq k^3 + 3k^2 + 3k + 1 \quad (\text{since } k \geq 4) \\
&= (k+1)^3.
\end{aligned}
$$

Hence The induction holds, and The statement is proven.

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b) Base Case $n = 1$.

$\cancel{3 = 3 \cdot 1} = 4^1 - 1 = 3 = 3 \cdot 1$

So 3 divides $4^1 - 1$.

Inductive Step. Let $k \geq 1$, and assume $3 \mid (4^k - 1)$.

Then
$$
\begin{aligned}
4^{k+1} - 1 &= 4 \cdot 4^k - 1 \\
&= 4(4^k - 1) + 3. \\
&= 4 \cdot (3q) + 3 \quad \text{by induction, for some } q. \\
&= 3(4q + 1).
\end{aligned}
$$

Hence The induction holds, and The statement is true.