MATH 374-001 Test 2

March 5, 2014

Name: Key

Directions: You have 50 minutes to complete the following exam. Show all applicable work. Answers without proper evidence of understanding will not receive credit.

1. (5 pts.) Prove or disprove the following statement.

If x and y are rational numbers, then the sum $x + y$ is a rational number.

Pf: Since x and y are Rational, we can write $x = \frac{a}{b}$, $y = \frac{c}{d}$ where a, b, c, d are integers, and b, d are non-zero.

Then $x+y = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

Since $ad+bc$ is an integer, and bd is a non-zero integer, $x+y$ is rational. ∎

2. (5 pts.) Prove or disprove the following statement.

For every natural number n, the number $2^n + 1$ is prime.

Counterexample: when $n=3$,

$$2^n+1 = 2^3+1 = 8+1 = 9 = 3 \cdot 3.$$

which is not prime.

3. (5 pts.) Prove or disprove the following statement.

For every integer n, if n^2 is an odd number, then n is an odd number.

By contrapositive:

If n is even, we can write $n = 2k$ for some integer k.

Then $n^2 = (2k)^2 = 2(2k^2)$. Since $2k^2$ is an integer,

n^2 is even.

Hence the contrapositive statement "If n is even, then n^2 is even" holds, so our original statement holds also.

4. (5 pts.) Write a recursive definition for the set S of all binary strings (strings of 0's and 1's) which do not contain the string 00. For example, the strings λ (the empty string), 0, 1, 01, 10, 11, 010, 011, 101, 110, 111, 0101, 0110, 1010, 0111, 1011, 1101, 1110, 1111 are all members of S, while 00, 001, 100 are not.

- The empty string λ is a member of S, and so is the string "0".

- If x and y are members of S, then so is $x1y$.

5. (5 pts.) The following program segment computes $n!$. Show that the program is correct by

(a) proving that $Q : \{y = (x-1)!\}$ is a loop invariant and

(b) evaluating the postcondition at loop termination.

$FACT(n)$:

$x = 2$
$y = 1$
while $x \neq n+1$ **do**
  $y = y * x$
  $x = x + 1$
end while
return y

a) Let x_i and y_i be the values of x and y after i iterations of the loop. We show Q is a loop invariant by induction.

Base: $i=0$.

$x_0 = 2$ and $y_0 = 1$. Since $1 = (2-1)!$, we see

That Q holds for $i=0$.

Induction: Assume $y_k = (x_k - 1)!$.

After the $k+1$st iteration,

we know that $y_{k+1} = y_k \cdot x_k$, and $\boxed{x_{k+1} = x_k + 1}$ $*$

Hence $y_{k+1} = y_k \cdot x_k$ (by the loop)

$= (x_k - 1)!\, x_k$ (by induction)

$= x_k!$ (by algebra)

$= (x_{k+1} - 1)!$ (by $*$).

So induction holds, and Q is a loop invariant.

b) At loop termination, we have $Q \wedge B'$, which says

$(y = (x-1)!) \wedge (x = n+1)$, which says $\boxed{y = n!}$

exactly as claimed.

6. (6 pts.) For the following recurrence relation, write the first 5 terms of the sequence, find a closed formula for $T(n)$, and prove the formula is true.

$$T(1) = 1$$
$$T(n) = 2T(n-1) + 1 \text{ for } n > 1.$$

$T(1) = 1 \qquad = 2 - 1 = 2^1 - 1$

$T(2) = 2(1) + 1 = 3 \qquad = 4 - 1 = 2^2 - 1$

$T(3) = 2 \cdot 3 + 1 = 7 \quad \ldots \quad = 8 - 1 = 2^3 - 1$

$T(4) = 2 \cdot 7 + 1 = 15 \qquad = 16 - 1 = 2^4 - 1$

$T(5) = 2 \cdot 15 + 1 = 31 \qquad = 32 - 1 = 2^5 - 1$

Claim: $T(n) = 2^n - 1$.

Pf: By induction on n.

Base: $n = 1$.

$T(1) = 1 = 2^1 - 1$. ✓

Inductive Step: Let $k \geq 1$, assume $T(k) = 2^k - 1$.

Then $T(k+1) = 2 \cdot T(k) + 1$ (by the recurrence)

$= 2 \cdot (2^k - 1) + 1$ (by induction)

$= 2^{k+1} - 2 + 1$

$= 2^{k+1} - 1$

Hence the inductive step holds, and our formula is true. ∎

Honor Statement:

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code. As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

Signature: ____________________