Math 374

Practice Test 2

Prove or Disprove The following Statements:

1) The sum of two odd integers is even

Pf: Let x and y be The integers. Since They are odd, there are integers k and n such That $x = 2k+1$ and $y = 2n+1$.

Hence $x+y = 2k+1 + 2n+1 = 2(k+n+1)$.

Since $k+n+1$ is an integer, $x+y$ is even ∎

2) The product of two irrational numbers is irrational.

Counterexample: $\sqrt{2}$ is irrational.

$\sqrt{2} \cdot \sqrt{2} = 2$ is The product of two irrational numbers,

but $2 \left(= \frac{2}{1}\right)$ is rational

3) The sum of a rational and an irrational is irrational.

Pf by Contradiction: Suppose x is rational, y is irrational, but for contradiction, That $x+y$ is rational. Then There exist $a, b \neq 0$, $m, n \neq 0$ such That $x = \frac{a}{b}$, $x+y = \frac{m}{n}$.

Then $y = \frac{m}{n} - \frac{a}{b} = \frac{mb-na}{nb}$. Since $(mb-na)$, $nb \neq 0$ are integers y is rational, contradicting That y is irrational. Hence our statement Holds.

4) Prove That $1 + 2^1 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 1$

for all $n \geq 1$.

By Induction:

Base: $n=1$, Says $1+2^1 = 2^2 - 1$, which is true.

Inductive: Assume $1+2^1+\dots+2^k = 2^{k+1} - 1$.

Then $\underbrace{1 + 2^1 + \dots + 2^k} + 2^{k+1} = \underbrace{2^{k+1} - 1} + 2^{k+1}$ by adding 2^{k+1} to both.

$= 2(2^{k+1}) - 1$

$= 2^{k+2} - 1$. Proving The statement for $k+1$. Hence our statement holds.

5) Prove That The following function correctly evaluates 2^n, by proving That $Q: \{ j = 2^i \}$ is a loop invariant, and evaluating The postcondition at loop termination.

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Power (n):   (positive integer)
   i = 1
   j = 2
   while i ≠ n do
       j = j*2
       i = i+1
   end while
   return j
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$Q(0)$ is true, since $j=2$ and $i=1$ before entering The loop, so $j = 2^i$.

Assuming $Q(k)$, we see That

$j_{k+1} = j_k \cdot 2$ by The loop assignment,

$= 2^{i_k} \cdot 2$ by $Q(k)$

$= 2^{i_k + 1}$ (algebra)

$= 2^{i_{k+1}}$ by The loop assignment.

Hence $Q(k+1)$ holds, So Q is an invariant.

At loop termination, we have $\{Q \wedge B'\}$. Hence we see $\{ j = 2^i \wedge i = n \}$, meaning $j = 2^n$, as claimed.

6) Give a recursive definition for all binary strings with an odd number of 0's.

Let S be our set of strings.

- 0 is in S.
- If x is in S, then $1x$ and $x1$ are in S.
- If x, y, z are in S, then xyz is in S.

7) Write the first 5 values of the sequence.

$$M(1) = 2$$
$$M(2) = 2$$
$$M(n) = 2(M(n-1)) + M(n-2) \quad \text{for } n > 2.$$

$M(1) = \underline{2}$

$M(2) = \underline{2}$

$M(3) = 2(2) + 2 = \underline{6}$

$M(4) = 2(6) + 2 = \underline{14}$

$M(5) = 2(14) + 6 = \underline{34}$.

8) Find a closed formula, and prove it holds, for the following recurrence relation.

$$F(1) = 2,$$
$$F(n) = 2F(n-1) + 2^n \quad \text{for } n>1.$$

$F(1) = 2$

$F(2) = 2(2) + 2^2 = 2\cdot 2^2$

$F(3) = 2(2^2) + 2^3 = 3\cdot 2^3$

$F(4) = 2\cdot(3\cdot 2^3) + 2^4 = 4\cdot 2^4$

$F(5) = 2(4\cdot 2^4) + 2^5 = 5\cdot 2^5$

$$\begin{aligned}
F(n) &= 2(F(n-1)) + 2^n \\
&= 2(2F(n-2) + 2^{n-1}) + 2^n \\
&= 2^2F(n-2) + 2\cdot 2^n \\
&= 2^2(2F(n-3) + 2^{n-2}) + 2\cdot 2^n \\
&= 2^3F(n-3) + 3\cdot 2^n \\
&\vdots \\
&= 2^k F(n-k) + k2^n \\
&\vdots \\
&= 2^{n-1}(F(1)) + (n-1)2^n \\
&= 2^n + (n-1)2^n = \underline{\underline{n\cdot 2^n}}.
\end{aligned}$$

<u>Prove:</u> <u>$F(n) = n2^n$ by induction</u>.

<u>Base</u>: $F(1) = 2 = 1\cdot 2^1$.

<u>Inductive</u>: Assume $F(k) = k2^k$.

Then $F(k+1) = 2F(k) + 2^{k+1}$ (by our <u>recurrence</u>)

$= 2(k2^k) + 2^{k+1}$ (by Inductive hypothesis)

$= k2^{k+1} + 2^{k+1}$ (algebra)

$= (k+1)2^{k+1}$ (algebra)

Hence our inductive step holds, and so we have

$F(n) = n2^n$ for all $n \geq 1$.