1) Show That $[(A \to B) \wedge (B \to C)] \to (A \to C)$ is a tautology.

A	B	C	$(A \to B)$	$(B \to C)$	$(A \to B) \wedge (B \to C)$	$(A \to C)$	$[(A \to B) \wedge (B \to C)] \to (A \to C)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T.

---

2) Show That $[(A \to B) \wedge (B \to C)] \to (A \to C)$ is valid using a proof sequence.

By deduction method, equivalent to $(A \to B) \wedge (B \to C) \wedge A \to C$.

1) $A \to B$ hyp
2) $B \to C$ hyp
3) A hyp
4) B 1,3, mp
5) C 2,4, mp.

3) Add explanations for the proof sequence of

$$A \wedge (B \to C) \wedge [(A \wedge B) \to (D \vee C')] \wedge B \to D.$$

1) A	hyp
2) $B \to C$	hyp
3) $(A \wedge B) \to (D \vee C')$	hyp
4) B	hyp
5) C	2, 4, MP
6) $A \wedge B$	1, 4, Conj
7) $D \vee C'$	3, 6, MP
8) $C \to D$	7, imp
9) D	5, 8, MP.

---

4) a) $(\forall x)(\forall y)[(B(x) \wedge S(y)) \to E(x,y)]$.

b) $(\exists x)(\forall y)[(B(x) \wedge S(y)) \to E(x,y)]$.

c) $(\forall x)(\forall y)[(S(y) \wedge E(x,y)) \to B(x)]$.

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5) Give a proof sequence for

$$(\exists x)P(x) \wedge (\forall x)(P(x) \to Q(x)) \to (\exists x)Q(x).$$

1) $(\exists x)P(x)$ hyp

2) $(\forall x)(P(x) \to Q(x))$ hyp

3) $P(a)$ 1, ei

4) $P(a) \to Q(a)$ 2, ui.

5) $Q(a)$ 3, 4, MP

6) $(\exists x)Q(x)$ 5, eg.

6) Give an interpretation to show The following is not valid.

$$(\forall x)\,[\,P(x) \Rightarrow Q(x)\,] \;\rightarrow\; [\,(\exists x)\,P(x) \rightarrow (\forall x)\,Q(x)\,]$$

<u>Domain:</u> All electronic devices.

$P(x)$ says "x is a Laptop."

$Q(x)$ says "x is a Computer."

Left is <u>true</u>: "For all electronics, if x is a laptop, Then x is a computer".

Right is <u>false</u>: "If There is at least one laptop, Then all electronic devices are computers".

Hence The <u>implication fails</u> (i.e. is <u>not valid</u>.)

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7) Precondition is $\{ x = a \wedge y = b \wedge z = c \}$. Postcondition is $\{ x = c \wedge y = a \wedge z = b \}$. Working from The bottom using assignment,

$\{ z = c \wedge x = a \wedge y = b \}$

temp = x

$\{ z = c \wedge temp = a \wedge y = b \}$

x = z

$\{ x = c \wedge temp = a \wedge y = b \}$

z = y

$\{ x = c \wedge temp = a \wedge z = b \}$

y = temp

$\{ x = c \wedge y = a \wedge z = b \}$

So The program segment is correct.

8) To show Correctness, we must check two Things.

$\{x=7 \wedge x \geq 0\}\ y=x\ \{y=7\}$

By assignment, we know

$\{x=7\}$
$y=x$
$\{y=7\}$

is true.

But $\{x=7\} \Longleftrightarrow \{x=7 \wedge x \geq 0\}$

So This implication is true.

$\{x=7 \wedge x<0\}\ y=2x\ \{y=7\}$.

This is true, because The Precondition is False.

($(P \to Q)$ is true if P is false)

9) (Typo... Counter example $1+1=2$, disproving The statement)

Instead Prove That The sum of 2 odd integers is even.

Let x, y be The two integers. Since each is odd, we can write $x = 2m+1$, $y = 2n+1$ for integers m, n.

Then
$$x+y = 2m+1+2n+1 = 2(m+n+1).$$

Hence $x+y$ is even.

10) Prove That The sum of an integer and its square is even.

Pf. Let n be an integer. Then

$$n + n^2 = n(n+1).$$

Since n and $(n+1)$ are consecutive, ~~at~~ one of Them is even. Since The product of an even and an odd is even, $n^2+n = n(n+1)$ is even.

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11) a) If G is not Eulerian, Then some vertex of G must have odd degree.

b) If G is Eulerian, Then Every vertex of G has even degree.

---

12) Prove That The sum of two rational numbers is Rational.

Pf. Let x, y be The numbers. Then we can write

$x = \frac{a}{b}$, $y = \frac{c}{d}$ for some integers a, b, c, d, with $b, d \neq 0$.

$$\text{Then } x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd}$$

$$= \frac{ad+cb}{bd}.$$

Since $(ad+cb)$ and (bd) are integers, $x+y$ is a rational number.

13) a) For $n \geq 1$, $7^n - 2^n$ is divisible by 5.

Pf. Base Case $n=1$. $7^1 - 2^1 = 5 = 5 \cdot 1$

So the statement holds for $n=1$.

Inductive step: Let $k \geq 1$. Assume $7^k - 2^k = 5m$ for some m.

Then $7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k$

$= 5 \cdot 7^k + 2 \cdot 7^k - 2 \cdot 2^k$

$= 5 \cdot 7^k + 2(7^k - 2^k)$

$= 5 \cdot 7^k + 2 \cdot (5m)$ (By I.H.)

$= 5(7^k + 2m)$.

Hence $7^{k+1} - 2^{k+1}$ is divisible by 5, completing the induction, and proving the statement.

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b) For $n \geq 7$, $n^2 > 5n + 10$.

Pf. Base Case $n=7$. $7^2 = 49 > 45 = 7 \cdot 5 + 10$.

So the statement holds for $n=7$.

Inductive Step: Let $k \geq 7$, and assume $k^2 > 5k + 10$.

Then $(k+1)^2 = k^2 + 2k + 1$

$> (5k+10) + (2k+1)$ (by I H).

$> (5k+10) + 5$ ($2k+1 > 5$ since $k \geq 7$)

$= 5(k+1) + 10$.

Hence the inductive step holds, and so the statement is true.

13 c) For all $n \geq 1$, $1+5+9+\ldots+(4(n)-3) = n(2n-1)$.

Pf: Base Case $n=1$:

$$1 = 1(2\cdot 1-1) = 1.$$

So the statement holds for $n=1$.

Inductive step: (Let $k \geq 1$) Assume $1+5+\ldots+(4k-3) = k(2k-1)$.

Then

$$1+5+\ldots+(4k-3)+(4(k+1)-3)$$
$$= k(2k-1) + 4(k+1)-3 \quad \text{(by I H)}.$$
$$= 2k^2+3k+1$$
$$= (k+1)(2k+1)$$
$$= (k+1)(2(k+1)-1).$$

Hence the inductive step holds, and so the statement is true.

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14)

$$\begin{array}{r} 8 \\ 90\overline{)\,735} \\ \underline{720} \\ 15 \end{array} \rightarrow \begin{array}{r} 6 \\ 15\overline{)\,90} \\ \underline{90} \\ 0 \end{array}$$

Since 15 was the remainder before zero,

$\text{GCD}(735, 90) = 15$.

$$\left(\begin{array}{l} 735 = 3\cdot 5\cdot 7^2 \\ 90 = 2\cdot 3^2\cdot 5 \end{array}\right)$$