EXAM #3 FOR MATH 514

Student's Name: ______________________________.

ID# (last four digits): _____________________.

Instructions: Each problem is worth the same amount. Show all your work for full credit. Use the backs of these pages if you need more room. Calculators are not allowed.

Problem #1: Define the following:

- An American style call option with strike price 100 and expiration time of 1 month hence on stock A.

  A contract that gives its owner (the one who bought the contract) the right to buy a share of the stock A (from the one who sold him the contract) for 100 at any time between the purchase time of the contract and one month thereafter.

- A European style put option with strike price 80 and exercise time of 1 month hence on stock B.

  A contract that gives its owner (the one who bought the contract) the right to sell a share of the stock B (to the one who sold him the contract) for 80 at a time exactly one month after the purchase time of the contract.

Problem #2: State the Generalized Law of One Price.

Consider two investment strategies, I_1 and I_2. Suppose that the (net) costs to initiate these strategies at time 0 are C_1 and C_2 respectively, where $C_1 < C_2$. Suppose $P_1(\omega)$ and $P_2(\omega)$ are the (net present value) payoffs for I_1 and I_2 respectively which occur in the future, where $\omega \in \Omega$ denotes a possible time behavior of the stocks involved. Suppose $P_1(\omega) \geq P_2(\omega)$ for all $\omega \in \Omega$. Then there is an arbitrage.

Problem #3: Prove the Generalized Law of One Price.

Suppose one buys I_1 and sells I_2. This produces a time 0 income of $C_2 - C_1$. Since one owns I_1 one can expect a future income (which when reduced to its present value) $P_1(\omega)$. Since one sold I_2 one can expect to have to pay (PV) $P_2(\omega)$ to its owner in the future. Thus the present value of the overall transaction is $C_2 - C_1 + P_1(\omega) - P_2(\omega) \geq C_2 - C_1 > 0$ for all $\omega \in \Omega$. This is an arbitrage.

Problem #4: State the Put-Call Option Parity Formula. Explain the meaning of all the symbols in the formula.

$S + P - C = Ke^{-rt}$, where S is the price of the stock at time 0, P is the price of a European-style put option with strike price K and exercise time t, and C is the price of a European-style call option with the same strike price and exercise time.

Problem #5: Use the Law of One Price to prove the put-call option parity formula.

Let I_1 denote the strategy of buying one share of stock and selling one (K, t) call option to party A, all at time 0. This incurs a net cost $C_1 = S - C$. Let I_2 denote the strategy of depositing Ke^{-rt} into a bank (earning continuously compounded interest at the rate r), as well as selling one (K, t) put option to party B, also at time 0. This incurs a net cost of $C_2 = Ke^{-rt} - P$. As is usual, let $P_1(\omega), P_2(\omega)$ denote the net present value payoffs of I_1 and I_2 respectively, where $\omega \in \Omega$ describes the behavior in time of the stock price, which for time t we denote by $S(t)(\omega)$. At time t the following takes place.

- Concerning I_1 if $S(t)(\omega) > K$ then party A will exercise the call option and buy the one share of stock from you (which you conveniently bought at time 0) for K, thus $P_1(\omega) = Ke^{-rt}$.
- Concerning I_1 if $S(t)(\omega) \leq K$ then party A will not exercise the call option and so you sell the stock for the price $S(t)(\omega)$, thus $P_2(\omega) = S(t)(\omega)e^{-rt}$.
- Concerning I_2 if $S(t)(\omega) \geq K$ then party B will not exercise the put option, but you may withdraw the amount K from the bank, thus $P_1(\omega) = Ke^{-rt}$.
- Concerning I_2 if $S(t)(\omega) < K$ then party B will exercise the put option and sell a share of stock to you for K, the exact amount which you may withdraw from the the bank. You sell the share for $S(t)(\omega)$, and thus $P_2(\omega) = S(t)(\omega)e^{-rt}$.

Comparing I_1 and I_2 we see that $P_1(\omega) = P_2(\omega)$ for all $\omega \in \Omega$. So by the Law of One Price either $C_1 = C_2$ or there is an arbitrage. Since there is no arbitrage, $S - C = C_1 = C_2 = Ke^{-rt} - P$, as we desired to prove.

Problem #6: Define:

- a function $f(x)$ is *convex*.

$f(x)$ is a real-valued function of the real variable x such that for every $x, y \in \text{domain}(f)$ and for every $0 \le \alpha \le 1$ we have $f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y)$.

- x^+.

$$x^+ = \begin{cases} x & x \ge 0 \\ 0 & x < 0 \end{cases}.$$

Problem #7: Suppose a stock has price $S(t)$ at time t. Give a formula for the payoff of a call option with strike price K and exercise time t. Explain why this formula works.

The payoff formula is $[S(t) - K]^+ = \begin{cases} S(t) - K & S(t) \ge K \\ 0 & S(t) < K \end{cases}$. If $S(t) \ge K$ then the option should be exercised, i.e. a share of stock will be bought for a cost K and then resold for the price $S(t)$, giving a net payoff of $S(t) - K$. If $S(t) < K$ then the option will not be exercised, and the net payoff is thus 0.

Problem #8: State the Arbitrage Theorem.

Suppose R is a $n \times m$ matrix, where R_{ij} is the payoff of a unit bet on wager i if the outcome of the experiment is j. Then exactly one of the following two alternatives is true.

(1) There is a $m \times 1$ probability vector $\mathbf{p}$ such that $R\mathbf{p} = \mathbf{0}$. (I.e. each wager has 0 expected payoff with respect to the probability distribution $\mathbf{p}$.)
(2) There is a $1 \times n$ betting strategy vector $\mathbf{x}$ such that $\mathbf{x}R > \mathbf{0}^T$. (I.e. $\mathbf{x}$ determines an arbitrage, since the jth component of $\mathbf{x}R$ is the total payoff of the strategy $\mathbf{x}$ if the outcome of the experiment is j, and this payoff is positive.)

Problem #9: Suppose a stock's current price is 100, and after 1 month its price could be either 50 or 200. Use the Arbitrage Theorem to find the price C of a call option with strike price 150 and exercise time of 1 month if the monthly interest rate is r.

We have two 'wagers' or investment strategies:

(1) Buy one share of the stock at time 0 and then sell it at time 1.
(2) Buy one call option with $(K, t) = (150, 1)$ at time 0 and then exercise the option at time 1 if it is advantageous to do so.

The 'experiment' is the market fluctuation between time 0 and time 1, with two outcomes:

(1) $S(1) = 200$.
(2) $S(1) = 50$.

This leads to a payoff matrix $R = \begin{pmatrix} -100 + \frac{200}{1+r} & -100 + \frac{50}{1+r} \\ -C + \frac{50}{1+r} & -C \end{pmatrix}$. $R_{21} = -C + \frac{50}{1+r}$ because the option costs C and when it is exercised one buys the stock for 150 and then resells it for 200, clearing 50 at time 1. Let $\mathbf{p} = \begin{pmatrix} p_1 \\ 1 - p_1 \end{pmatrix}$. Then the first row of $R\mathbf{p} = \mathbf{0}$ is

$$\begin{aligned} \left(-100 + \frac{200}{1+r}\right) p_1 + \left(-100 + \frac{50}{1+r}\right)(1 - p_1) &= 0 \\ -100p_1 + \frac{200p_1}{1+r} - 100(1 - p_1) + \frac{50}{1+r} - \frac{50p_1}{1+r} &= 0 \\ -100 + \frac{50}{1+r} &= -\frac{150p_1}{1+r} \\ -2(1+r) + 1 &= -3p_1 \\ \frac{1+2r}{3} &= p_1 \end{aligned}$$

In a similar way (using this result) the second row of $R\mathbf{p} = \mathbf{0}$ yields $C = \frac{50(1+2r)}{3(1+r)} = \frac{1}{3}\left(100 - \frac{50}{1+r}\right)$. This is the no-arbitrage option price we are looking for.

Problem #10: In the situation of the previous problem explain what happens financially to both the person selling the option and the person buying the option if the option is sold at a price $C' < C$, where C is the no-arbitrage price.

If the option price is C' instead of C (found in the previous problem) then the payoff matrix becomes $R' = \begin{pmatrix} -100 + \frac{200}{1+r} & -100 + \frac{50}{1+r} \\ -C' + \frac{50}{1+r} & -C' \end{pmatrix}$. Because $\mathbf{p}$ was uniquely determined by the first row of $R\mathbf{p} = \mathbf{0}$, and C was uniquely determined by the second row of $R\mathbf{p} = \mathbf{0}$, we will now have $R'\mathbf{p} \neq \mathbf{0}$ for any probability vector $\mathbf{p}$. Hence by the Arbitrage Theorem there must exist a betting strategy vector $\mathbf{x}$ such that $\mathbf{x}R' > \mathbf{0}$, i.e. which determines an arbitrage. Define $\mathbf{x} = (-1, 3)$. Then

$$\begin{aligned} \mathbf{x}R' &= \begin{pmatrix} -1 & 3 \end{pmatrix} \begin{pmatrix} -100 + \frac{200}{1+r} & -100 + \frac{50}{1+r} \\ -C' + \frac{50}{1+r} & -C' \end{pmatrix} \\ &= \begin{pmatrix} 100 - \frac{200}{1+r} - 3C' + \frac{150}{1+r} & 100 - \frac{150}{1+r} - 3C' \end{pmatrix} \\ &= \begin{pmatrix} 3(C - C') & 3(C - C') \end{pmatrix}. \end{aligned}$$

Since $C' < C$, both entries of $\mathbf{x}R'$ are positive, showing that $\mathbf{x} = (-1, 3)$ (i.e. sell stock at time 0 and buy it back at time 1 and buy 3 call options at time 0 and exercise them at time 1 if warranted) yields an arbitrage for the option buyer, who clears $3(C - C')$ no matter what happens to the stock.

The point of this problem is to explain out of whose pocket this money comes. So let us examine the financial outcome for the seller of the options. The present value of the option seller's net gain is $3C' - \frac{3}{1+r}[S(1) - 150]^+$, since when he sells the 3 options he gains $3C'$ and when (if) they are exercised he (may have to) pay out $3[S(1)-150]^+$ at time 1. He may gain or lose money overall, depending on what the stock price does between time 0 and time 1. But he would have gained more or lost less if he had charged C instead of C' for the options. No matter what happens to the stock the options seller is $3(C - C')$ worse off if he sells the options for C' instead of for C. Thus the money gained in the options buyer's arbitrage comes out of the options seller's pocket. Hence the option seller has a financial incentive to charge the no-arbitrage price C for the options.