Homework Solutions

5.103 Y_1, Y_2, Y_3 are RVs

$$\begin{pmatrix} E(Y_1) \\ E(Y_2) \\ E(Y_3) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \qquad \begin{pmatrix} V(Y_1) & Cov(Y_1, Y_2) & Cov(Y_1, Y_3) \\ & V(Y_2) & Cov(Y_2, Y_3) \\ & & V(Y_3) \end{pmatrix} = \begin{pmatrix} 4 & 1 & -1 \\ & 6 & 0 \\ & & 8 \end{pmatrix}$$

Find $E(3Y_1 + 4Y_2 - 6Y_3) = 3E(Y_1) + 4E(Y_2) - 6E(Y_3)$

$$= 3 \cdot 2 + 4(-1) - 6 \cdot 4 = -22$$

Find $V(3Y_1 + 4Y_2 - 6Y_3)$

$$= (3 \quad 4 \quad -6) \begin{pmatrix} 4 & 1 & -1 \\ 1 & 6 & 0 \\ -1 & 0 & 8 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix} = (3 \quad 4 \quad -6) \begin{pmatrix} 22 \\ 27 \\ -51 \end{pmatrix}$$

$$= 480$$

To see why this works let $U = c_1 Y_1 + c_2 Y_2 + c_3 Y_3$

$$V(U) = Cov(U, U) = Cov(c_1 Y_1 + c_2 Y_2 + c_3 Y_3, U)$$

$$= c_1 Cov(Y_1, U) + c_2 Cov(Y_2, U) + c_3 Cov(Y_3, U)$$

$$= (c_1 \quad c_2 \quad c_3) \begin{pmatrix} Cov(Y_1, U) \\ Cov(Y_2, U) \\ Cov(Y_3, U) \end{pmatrix}$$

$$= (c_1 \quad c_2 \quad c_3) \begin{pmatrix} Cov(Y_1, c_1 Y_1 + c_2 Y_2 + c_3 Y_3) \\ Cov(Y_2, c_1 Y_1 + c_2 Y_2 + c_3 Y_3) \\ Cov(Y_3, c_1 Y_1 + c_2 Y_2 + c_3 Y_3) \end{pmatrix}$$

$$= (c_1 \quad c_2 \quad c_3) \begin{pmatrix} Cov(Y_1, Y_1) c_1 + Cov(Y_1, Y_2) c_2 + Cov(Y_1, Y_3) c_3 \\ Cov(Y_2, Y_1) c_1 + Cov(Y_2, Y_2) c_2 + Cov(Y_2, Y_3) c_3 \\ Cov(Y_3, Y_1) c_1 + Cov(Y_3, Y_2) c_2 + Cov(Y_3, Y_3) c_3 \end{pmatrix}$$

$$= (c_1 \quad c_2 \quad c_3) \begin{pmatrix} Cov(Y_1, Y_1) & Cov(Y_1, Y_2) & Cov(Y_1, Y_3) \\ Cov(Y_2, Y_1) & Cov(Y_2, Y_2) & Cov(Y_2, Y_3) \\ Cov(Y_3, Y_1) & Cov(Y_3, Y_2) & Cov(Y_3, Y_3) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

5.129 Suppose Y_1, Y_2 have a bivariate normal distribution. Show that $f(y_1 \mid y_2)$ is normal with mean $\mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y_2 - \mu_2)$ and variance $\sigma_1^2(1-\rho^2)$.

First we compute $f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1$. Let $z_1 = \frac{y_1 - \mu_1}{\sigma_1}$ and $z_2 = \frac{y_2 - \mu_2}{\sigma_2}$. $f(y_1, y_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[z_1^2 - 2\rho z_1 z_2 + z_2^2 \right] \right\}$

$$f_2(y_2) = \frac{1}{2\pi \sigma_2 \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[z_1^2 - 2\rho z_1 z_2 + z_2^2 \right] \right\} dz_1 \quad \left(\text{since } dz_1 = \frac{dy_1}{\sigma_1}\right)$$

$$= \frac{1}{2\pi \sigma_2 \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[z_1^2 - 2\rho z_1 z_2 + \rho^2 z_2^2 + (1-\rho^2) z_2^2 \right] \right\} dz_1$$

$$= \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{1}{2} z_2^2} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[z_1 - \rho z_2 \right]^2 \right\} dz_1 \qquad u = \frac{z_1 - \rho z_2}{\sqrt{1-\rho^2}}$$

$$= \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{1}{2} z_2^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2} u^2 \right\} du = \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{1}{2} z_2^2}$$

Second $f(y_1 \mid y_2) = \frac{f(y_1, y_2)}{f_2(y_2)}$

$$= \sigma_2 \sqrt{2\pi}\, e^{\frac{1}{2} z_2^2} \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[(z_1 - \rho z_2)^2 + (1-\rho^2) z_2^2 \right] \right\}$$

$$= \frac{1}{\sigma_1 \sqrt{2\pi} \sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2} \left[\frac{(z_1 - \rho z_2)^2}{1-\rho^2} \right] \right\}$$

$$= \frac{1}{\sqrt{2\pi}\, \sigma_1 \sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2} \left[\frac{y_1 - \mu_1 - \rho \sigma_1 z_2}{\sigma_1 \sqrt{1-\rho^2}} \right]^2 \right\}$$

normal density with mean $\mu_1 + \rho \sigma_1 z_2$, variance $\sigma_1^2(1-\rho^2)$.