Solutions to Homework

$\boxed{2.7}$ Let M be the set of the three men and W the set of the two women

$M \cup W$ is the set of the five applicants

Since the two jobs are identical there is no need to indicate which of the two jobs an applicant is assigned to. Thus an outcome of the selection process is a subset of $M \cup W$ consisting of two people. Thus

$$S = \binom{M \cup W}{2} \qquad |S| = \binom{5}{2} = \frac{5 \cdot 4}{2} = 10$$

$$A = \{ s \in S \mid s \subset M \} \qquad |A| = \binom{3}{2} = 3$$

$$B = \{ s \in S \mid s \cap W \neq \emptyset \} = \bar{A} \qquad |B| = |S| - |A| = 10 - 3 = 7$$

List the outcomes in the following events:

$$A = \{ \{M_1, M_2\}, \{M_1, M_3\}, \{M_2, M_3\} \} = \bar{B}$$

$$A \cup B = S = \{ \{M_1, M_2\}, \{M_2, M_3\}, \{M_3, M_1\}, \{M_1, W_1\}, \{M_2, W_1\}, \{M_3, W_1\}, \{M_1, W_2\}, \{M_2, W_2\}, \{M_3, W_2\}, \{W_1, W_2\} \}$$

$A \cap B = \emptyset = \{\}$, $A \cap \bar{B} = A$ (listed above).

$\boxed{2.11}$ $P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5) = 1$

(a) $0.15 + 0.15 + 0.4 + 2p + p = 1$

$3p = 1 - 0.7 = .3$

$p = P(E_5) = .1$. Also $P(E_4) = 2(.1) = .2$

(b) $.3 + .1 + q + q + q = 1$

$3q = 1 - .4 = .6$

$P(E_3) = P(E_4) = P(E_5) = q = .2$

$\boxed{2.19}$ $V = \{ v_1, v_2, v_3 \}$ = set of Vendors

(a) $S = V \times V = \{ (v_1,v_1), (v_1,v_2), (v_1,v_3),$
$(v_2,v_1), (v_2,v_2), (v_2,v_3),$
$(v_3,v_1), (v_3,v_2), (v_3,v_3) \}$

(b) $|S| = |V||V| = 3 \cdot 3 = 9$ $\quad P(\{(v_i,v_j)\}) = \frac{1}{9}$ $\quad 1 \le i,j \le 3$

(c) A = same vendor gets both orders
$= \{ (v_1,v_1), (v_2,v_2), (v_3,v_3) \}$ $\quad P(A) = \frac{3}{9} = \frac{1}{3}$

$B = v_2$ gets at least one order
$= \{ (v_1,v_2), (v_2,v_1), (v_2,v_2), (v_2,v_3), (v_3,v_2) \}$ $\quad P(B) = \frac{5}{9}$

$A \cap B = \{ (v_2,v_2) \}$ $\quad P(A \cap B) = \frac{1}{9}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{5}{9} - \frac{1}{9} = \frac{7}{9}$

$\boxed{2.28}$ $M = \{ \overset{m}{\text{minority}} \}$ $\quad N = \{ p_1, p_2, p_3 \}$ = other non minority applic.

(a) $S = \binom{M \cup N}{2} = \{ \{m,p_1\}, \{m,p_2\}, \{m,p_3\},$
$\{p_1,p_2\}, \{p_2,p_3\}, \{p_3,p_1\} \}$

(b) $|S| = \binom{4}{2} = 6$ $\quad P(\{s\}) = \frac{1}{6}$ for every $s \in S$

(c) E = the minority applicant is selected for one of the positions
$= \{ \{m,p_1\}, \{m,p_2\}, \{m,p_3\} \}$ $\quad P(E) = \frac{3}{6} = \frac{1}{2}$

$\boxed{2.34}$ P = patients $= \{p_1, p_2, p_3\}$. $\quad T$ = stations $= \{s_1, s_2, s_3\}$.

An assignment of patients to stations is a mapping $f: P \to T$

(a) $S = T^P$. Denote each $f \in T^P$ by the 3-tuple $(f(p_1), f(p_2), f(p_3))$

$S = \{ (s_1,s_1,s_1), (s_1,s_1,s_2), (s_1,s_1,s_3), (s_1,s_2,s_1), (s_1,s_2,s_2), (s_1,s_2,s_3),$
$(s_1,s_3,s_1), (s_1,s_3,s_2), (s_1,s_3,s_3), (s_2,s_1,s_1), (s_2,s_1,s_2), (s_2,s_1,s_3),$
$(s_2,s_2,s_1), (s_2,s_2,s_2), (s_2,s_2,s_3), (s_2,s_3,s_1), (s_2,s_3,s_2), (s_2,s_3,s_3),$
$(s_3,s_1,s_1), (s_3,s_1,s_2), (s_3,s_1,s_3), (s_3,s_2,s_1), (s_3,s_2,s_2), (s_3,s_2,s_3),$
$(s_3,s_3,s_1), (s_3,s_3,s_2), (s_3,s_3,s_3) \}$

(b) $A = \text{surj}(P,T) = \text{bij}(P,T)$ = each station received a patient

$A = \{ (s_1, s_2, s_3), (s_1, s_3, s_2), (s_2, s_1, s_3), (s_2, s_3, s_1),$
$(s_3, s_1, s_2), (s_3, s_2, s_1) \}$

ⓒ $|S| = |T^P| = |T|^{|P|} = 3^3 = 27 \quad |A| = |bij(P,T)| = 3! = 6$

$P(A) = |A|/|S| = 6/27 = 2/9$

$\boxed{2.37}$ Let $\hat{6} = \{1,2,3,4,5,6\}$, C = set of six cities

Each itenerary can thought of as a 1-1, onto mapping

$f: C \to \hat{6} \qquad f(\text{city}) =$ order it is visited

ⓐ $S = bij(C, \hat{6}) \quad |S| = 6! = 720$

ⓑ E = Denver is visited before San Francisco

$E = \{ g \in S \mid g(\text{Denv}) < g(SF) \}$

Let F = other four cities $= C \setminus \{\text{Denv}, SF\}$.

Define $f: E \longrightarrow inj(F, \hat{6})$

$g \longmapsto (g|_F$, i.e. the mapping g restricted to $F)$

Claim: f is 1-1 and onto

to see: if $g_1, g_2 \in E$ and $f(g_1) = f(g_2)$, then $g_1|_F = g_2|_F$

$g_1(F) = g_2(F)$ is a 4 element subset of $\hat{6}$. Since

$g_1, g_2 \in bij(C, \hat{6}) \quad \{g_1(\text{Denv}), g_1(SF)\} = \{g_2(\text{Denv}), g_2(SF)\}$

$= \hat{6} \setminus g_1(F)$. Since $g_1, g_2 \in E \quad g_1(\text{Denv}) < g_1(SF)$

and $g_2(\text{Denv}) < g_2(SF)$. Thus g_1 & g_2 have the same values on $\{\text{Denv}, SF\}$ as well as on F. Thus $g_1 = g_2$.

This shows f is 1-1.

If $h \in inj(F, \hat{6})$, $\text{range}(h) = h(F) \subset \hat{6}$ has 4 elements. Let $\{i, j\} = \hat{6} \setminus h(F)$ where $i < j$. Define

$g(\text{Denv}) = i$, $g(SF) = j$, $g|_F = h$. Clearly $g \in E$ and

$f(g) = h$. Thus f is onto. $\therefore$ Claim proved

So $|E| = |inj(F, \hat{6})| = \dfrac{|\hat{6}|!}{(|\hat{6}| - |F|)!} = \dfrac{6!}{(6-4)!} = \dfrac{6!}{2}$

So $P(E) = \frac{|E|}{|S|} = \frac{6!}{2} \cdot \frac{1}{6!} = \frac{1}{2}$

$\boxed{J.42}$ T = ten new engineers. P = three distinct positions
Each way of filling the positions is an injective mapping $f: P \to T$. It is injective since two distinct positions cannot be filled with the same engineer. Thus
$S = inj(P, T)$ $\quad |S| = \frac{|T|!}{(|T| - |P|)!} = \frac{10!}{(10-3)!} = 10 \cdot 9 \cdot 8 = 720$

$\boxed{J.43}$ N = set of nine taxis $\tilde{A}$ = set of three airports $= \{A, B, C\}$
Each mapping $\alpha: N \to \tilde{A}$ tells which airport each taxi is sent to.

$$|\{\alpha \in \tilde{A}^N \mid |\alpha^{-1}(\{A\})| = 3, |\alpha^{-1}(\{B\})| = 5, |\alpha^{-1}(\{C\})| = 1\}|$$

$$= \binom{9}{3\ \ 5\ \ 1} = \frac{9!}{3!\,5!\,1!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{3 \cdot 2} = 504$$

as claimed in class