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Coefficient quantization for frames in Banach spaces ☆

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ABSTRACT

Let (e_i) be a fundamental system of a Banach space. We consider the problem of approximating linear combinations of elements of this system by linear combinations using quantized coefficients. We will concentrate on systems which are possibly redundant. Our model for this situation will be frames in Banach spaces.

1. Introduction

Hilbert space frames provide a crucial theoretical underpinning for compression, storage and transmission of signals because they provide robust and stable representation of vectors. They also have applications in mathematics and engineering in a wide variety of areas including sampling theory [1], operator theory [17], harmonic analysis [15], nonlinear sparse approximation [9], pseudo-differential operators [16], and quantum computing [10].

In many situations it is useful to think of a signal as being a vector x in a Hilbert space and being represented as a (finite or infinite) sequence $(\langle x_i, x\rangle)_{i=1}^{\infty}$, where (x_i) is a *frame*, i.e. a sequence in H which satisfies for some $0 < a \leqslant b$,

$$a\|x\|^2 \leqslant \sum \big|\langle x_i, x\rangle\big|^2 \leqslant b\|x\|^2, \quad \text{whenever } x \in H. \tag{1}$$

Since the sequence (x_i) does not have to be (and usually is not) a basis for H, the representation of an $x \in H$ as the sequence $(\langle x_i, x\rangle)_{i=1}^{\infty}$ includes some redundancy, which, for example, can be used to correct errors in transmissions [13]. Using a Hilbert space as the underlying space has, *inter alia*, the advantage of an easy reconstruction formula. Nevertheless, there are circumstances which make it necessary to leave the confines of a Hilbert space, and generalize frames to the category of Banach spaces. One such instance occurs when we wish to replace the frame coefficients by *quantized coefficients*, i.e. by integer multiples of a given $\delta > 0$.

An example of such a situation is described by Daubechies and DeVore in [6]: Let $f \in L_2(-\infty, \infty)$ be a *band-limited function*, to wit, the support of the Fourier transform $\hat{f}$ is contained in $[-\Omega, \Omega]$ for some $\Omega > 0$. For simplicity we assume

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that $\Omega=\pi$. Now we can think of $\hat f$ as an element of $L_2[-\pi,\pi]$, write $\hat f$ on $[-\pi,\pi]$ as a series in e^{-inx}, $n\in\mathbb{Z}$, and apply the inversion formula for the Fourier transform. This leads to the *sampling formula*

$$f(x)=\sum_{n\in\mathbb{Z}} f(n)\frac{\sin(x\pi-n\pi)}{x\pi-n\pi},\quad x\in\mathbb{R}.$$

This series converges 'badly.' In particular it is not absolutely convergent in general. Therefore we consider some $\lambda>1$ and think of the space $L_2[-\pi,\pi]$ as being embedded (in the natural way) into $L_2[-\lambda\pi,\lambda\pi]$. The family of functions $(e^{-inx/\lambda})_{n\in\mathbb{Z}}$ forms an orthogonal basis for $L_2[-\lambda\pi,\lambda\pi]$, and it can be viewed as a frame for the 'smaller' space $L_2[-\pi,\pi]$ (see Section 2). We write $\hat f(\xi)=\sqrt{2\pi}\,\hat\rho(\xi)\cdot\hat f(\xi)$, where $\hat\rho:\mathbb{R}\to[0,1/\sqrt{2\pi}]$ is $\mathcal{C}_\infty$, $\hat\rho|_{[-\pi,\pi]}\equiv 1/\sqrt{2\pi}$, and $\hat\rho|_{(-\infty,\lambda\pi]\cup[\lambda\pi,\infty)}\equiv 0$. Now we can express $\hat f$ on $[-\lambda\pi,\lambda\pi]$ as a series in $(e^{-inx/\lambda})$ and apply the inverse transform once again. This leads to the expansion

$$f(x)=\frac{1}{\lambda}\sum_{n\in\mathbb{Z}} f\left(\frac{n}{\lambda}\right)\rho\left(x-\frac{n}{\lambda}\right),\quad x\in\mathbb{R},$$

which not only converges faster, but is also absolutely unconditionally convergent, since $\hat\rho$ is C_∞, and, thus, ρ and all its derivatives are in $L_1(\mathbb{R})$.

Now assume that $\|f\|_{L_\infty}\leqslant 1$ (note that bandlimited functions are bounded in L_∞). It was shown in [6] that the $\Sigma-\Delta$-*quantization algorithm* can be used to find a sequence $(q_n)_{n\in\mathbb{Z}}\subset\{-1,1\}$ for which

$$\left|f(x)-\frac{1}{\lambda}\sum_{n\in\mathbb{Z}} q_n\rho\left(x-\frac{n}{\lambda}\right)\right|\leqslant\frac{1}{\lambda}\|\rho'\|_{L_1},\quad \text{for } x\in\mathbb{R}.$$

This means that our approximation does not hold in L_2 (and it need not for the applications at hand) but it does hold in the Banach space L_∞ (in fact in $C(\mathbb{R})$).

We consider therefore a signal to be an arbitrary vector x in a Banach space X and ask if there is a dictionary (e_i), e.g. some sequence (e_i) whose span is dense in X, so that x can be approximated in norm, up to some $\varepsilon>0$, by a linear combination of the e_i's using only coefficients from a discrete alphabet, i.e. the integer multiples of some given δ. The case that (e_i) is a non-redundant system, for example a basis, or, more generally, a total fundamental minimal system, was treated in [8]. It was shown there, for example, that if (e_i) is a semi-normalized fundamental and total minimal system which has the property that for some $\varepsilon,\delta>0$ every vector of the form $x=\sum_{i\in E}a_ie_i$, with $E\subset\mathbb{N}$ finite, can be ε-approximated by a vector $\tilde x=\sum_{i\in E}\delta k_ie_i$, with $(k_i)\subset\mathbb{Z}$, then (e_i) must have a subsequence which is either equivalent to unit-vector basis of c_0, or to the summing basis for c_0. Conversely, every separable Banach space X containing c_0 admits such a total fundamental minimal system.

In this work we will concentrate on redundant dictionaries. Our model for redundant dictionaries will be frames in Banach spaces. In Section 2 we shall recall their definition and make some elementary observations. Before we tackle the problem of coefficient quantization with respect to frames, we first have to ask ourselves what exactly we mean by a *meaningful coefficient quantization*. In Section 3 we recall the notion *Net Quantization Property* (NQP) as introduced for fundamental systems in [8]. We shall then present several examples of systems which formally satisfy the NQP, but on the other hand clearly do not accomplish the goals of quantization, namely data compression and easy reconstruction. These examples will lead us to a notion of quantization which is more restrictive, and more meaningful, in the case of redundant systems.

In Section 4 we ask under which circumstances one can approximate a vector in a Banach space X by a vector with quantized coefficients which are bounded in some *associated sequence space* Z with a basis (z_i) (see Definition 4.1). If (z_i) has nontrivial lower estimates this is only possible if one reconciles with the fact that the length of the frame increases exponentially with the dimension of the underlying space. We shall show this type of quantization cannot happen if (z_i) satisfies nontrivial lower and upper estimates. The proof of these facts utilizes volume arguments and must therefore be formulated first in the finite dimensional case. An infinite dimensional argument proves directly that the associated space Z with a semi-normalized basis (z_i) cannot be reflexive. In particular, there is no semi-normalized frame (x_i) for an infinite dimensional Hilbert space so that for some choice of $0<\varepsilon,\delta<1$ and $C\geqslant 1$, every $x\in H$, $\|x\|=1$, can be ε-approximated by a vector $\tilde x=\sum\delta k_ix_i$, with $(k_i)\subset\mathbb{Z}$ and $\sum\delta^2k_i^2\leqslant C$.

In Section 5 we consider conditions under which an n-dimensional space admits, for given $\varepsilon,\delta>0$ and $C\geqslant 1$, a finite frame $(x_i)_{i=1}^N$, so that every element in the *zonotope* $\{\sum_{i=1}^N a_ix_i:\ |a_i|\leqslant 1\}$ can be ε-approximated by some element from $\{\sum_{i=1}^N\delta k_ix_i:\ k_i\in\mathbb{Z},\ |k_i|\leqslant C/\delta\}$. Using results from convex geometry we shall show that this is only possible for spaces X with trivial cotype. Among others, we provide an answer to a question raised in [8] and prove that ℓ_1 does not have a semi-normalized basis with the NQP.

In the final section we will state some open problems.

All Banach spaces are considered to be spaces over the real field $\mathbb{R}$. S_X and B_X, denote the unit sphere and the unit ball of a Banach space X, respectively. For a set S we denote by $c_{00}(S)$, or simply c_{00}, if $S=\mathbb{N}$, the set of all families $x=(\xi_s)_{s\in S}$ with finite support, $\mathrm{supp}(x)=\{s\in S:\ \xi_s\neq 0\}$. The unit vector basis of c_{00}, as well as the unit vector basis of ℓ_p, $1\leqslant p<\infty$, and c_0 is denoted by (e_i).

A *Schauder basis*, or simply a *basis*, of a Banach space X is a sequence (x_n), which has the property that every x can be uniquely written as a norm converging series $x = \sum a_i x_i$. It follows then from the Uniform Boundedness Principle that the coordinate functionals (x_n^*), $x_n^* : X \to \mathbb{R}$, $\sum a_i x_i \mapsto a_n$ are bounded (cf. [11]) and the projections P_n, with

$$P_n : X \to X, \qquad x = \sum a_i x_i \mapsto \sum_{i=1}^{n} a_i x_i, \quad \text{for } n \in \mathbb{N}$$

are continuous and uniformly bounded in the operator norm. We call $C = \sup_{n\in\mathbb{N}} \|P_n\|$ the *basis constant of* (x_i) and $K = \sup_{0\leqslant m\leqslant n} \|P_n - P_m\|$ $(P_0 \equiv 0)$ the *projection constant of* (x_i). Note that $C \leqslant K \leqslant 2C$. We call (x_n) *monotone* if $C = 1$ and *bimonotone* if also $K = 1$. A basis (x_n) is called *unconditional* if for any $x \in X$ the unique representation $x = \sum a_n x_n$ converges unconditionally. This is equivalent (cf. [11]) to the property that for all $(a_i) \in c_{00}$

$$K_u = \sup\left\{ \left\| \sum \pm a_i x_i \right\| : \ \left\| \sum a_i x_i \right\| = 1 \right\} < \infty.$$

If X is a finite dimensional space we can represent it isometrically as $(\mathbb{R}^n, \|\cdot\|)$ where $\|\cdot\|$ is a norm function on $\mathbb{R}^n$. With this representation we consider the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ and denote it by $\mathrm{Vol}(A)$. Of course $\mathrm{Vol}(A)$ depends on the representation of X. Nevertheless, if we only consider certain ratios of volumes this is not the case. Therefore, the quotient $\mathrm{Vol}(A)/\mathrm{Vol}(B)$ is well defined even in abstract finite dimensional spaces without any specific representation.

2. Frames in Hilbert spaces and Banach spaces

In this section we give a short review of the concept of frames in Banach spaces, and make some preparatory observations. Let us start with the well known notion of Hilbert space frames.

Definition 2.1. Let H be a (finite or infinite dimensional) Hilbert space. A sequence $(x_j)_{j\in\mathbb{J}}$ in H, $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \ldots, N\}$, for some $N \in \mathbb{N}$, is called a *frame of H* or *Hilbert frame for H* if there are $0 < a \leqslant b < \infty$ so that

$$a\|x\|^2 \leqslant \sum_{j\in\mathbb{N}} \left|\langle x, x_j\rangle\right|^2 \leqslant b\|x\|^2 \quad \text{for all } x \in H. \tag{2}$$

For a frame $(x_j)_{j\in\mathbb{J}}$ of H we consider the operator

$$\Theta : H \to \ell_2(\mathbb{J}), \qquad x \mapsto \left(\langle x, x_j\rangle\right)_{j\in\mathbb{J}},$$

its adjoint

$$\Theta^* : \ell_2(\mathbb{J}) \to H, \qquad (\xi_j)_{j\in\mathbb{J}} \mapsto \sum_{j\in\mathbb{J}} \xi_j x_j$$

and their product

$$I = \Theta^* \circ \Theta : H \to H, \qquad x \mapsto \sum_{j\in\mathbb{J}} \langle x, x_j\rangle x_j.$$

Since

$$a\|x\|^2 \leqslant \sum_{j\in\mathbb{N}} \left|\langle x, x_j\rangle\right|^2 = \left\langle x, \sum_{j\in\mathbb{N}} \langle x, x_j\rangle x_j \right\rangle = \left\langle x, I(x)\right\rangle \leqslant b\|x\|^2,$$

I is a positive and invertible operator with $a\,\mathrm{Id}_H \leqslant I \leqslant b\,\mathrm{Id}_H$ and thus,

$$x = I^{-1} \circ I(x) = \sum_{j\in\mathbb{N}} \langle x, x_j\rangle I^{-1}(x_j), \quad \text{or}$$
$$x = I \circ I^{-1}(x) = \sum_{j\in\mathbb{N}} \left\langle I^{-1}(x), x_j\right\rangle x_j = \sum_{j\in\mathbb{N}} \left\langle x, I^{-1}(x_j)\right\rangle x_j.$$

For an introduction to the theory of Hilbert space frames we refer the reader to [2] and [4]. We follow [17] and [5] for the generalization of frames to Banach spaces.

Definition 2.2 *(Schauder frame).* Let X be a (finite or infinite dimensional) separable Banach space. A sequence $(x_j, f_j)_{j\in\mathbb{J}}$, with $(x_j)_{j\in\mathbb{J}} \subset X$, $(f_j)_{j\in\mathbb{J}} \subset X^*$, and $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \ldots, N\}$, for some $N \in \mathbb{N}$, is called a (*Schauder*) *frame of X* if for every $x \in X$

$$x = \sum_{j\in\mathbb{J}} f_j(x) x_j. \tag{3}$$

In case that $\mathbb{J}=\mathbb{N}$, we mean that the series in (3) converges in norm, i.e. that $x=\lim_{n\to\infty}\sum_{j=1}^n f_j(x)x_j$.

An *unconditional frame of* X is a frame $(x_i, f_i)_{i\in\mathbb{N}}$ for X for which the convergence in (3) is unconditional.

We call a frame (x_i, f_i) *bounded* if

$$\sup_i \|x_i\| < \infty \quad \text{and} \quad \sup_i \|f_i\| < \infty,$$

and semi-normalized if (x_i) and (f_i) are semi-normalized, i.e. if $0<\inf_i \|x_i\| \leqslant \sup_i \|x_i\| < \infty$ and $0 < \inf_i \|f_i\| \leqslant \sup_i \|f_i\| < \infty$.

In the following remark we make some easy observations.

Remark 2.3. Let $(x_i, f_i)_{i\in\mathbb{N}}$ be a frame of X.

(a) If $\inf_{i\in\mathbb{N}}\|x_i\|>0$, then $f_i \xrightarrow{w^*} 0$ as $i\to\infty$.

(b) Using the Uniform Boundedness Principle we deduce that

$$K=\sup_{x\in B_X}\sup_{m\leqslant n}\left\|\sum_{i=m}^n f_i(x)x_i\right\|<\infty.$$

This implies that if $\inf_{i\in\mathbb{N}}\|x_i\|>0$ then (f_i) is bounded and if $\inf_{i\in\mathbb{N}}\|f_i\|>0$ then (x_i) is bounded.

We call K the *projection constant of* (x_i, f_i). The projection constant for finite frames is defined accordingly.

(c) For all $f\in X^*$ and $x\in X$ it follows that

$$f(x)=f\left(\sum_{i=1}^\infty f_i(x)x_i\right)=\sum_{i=1}^\infty f_i(x)f(x_i)=\lim_{n\to\infty}\left(\sum_{i=1}^n f(x_i)f_i\right)(x),$$

and, thus,

$$f=w^*-\sum_{i=1}^\infty f(x_i)f_i.$$

Moreover, for $m\leqslant n$ in $\mathbb{N}$ it follows that

$$\left\|\sum_{i=m}^n f(x_i)f_i\right\|=\sup_{x\in B_X}\left|\sum_{i=m}^n f(x_i)f_i(x)\right|\leqslant\|f\|\sup_{x\in B_X}\left\|\sum_{i=m}^n f_i(x)x_i\right\|\leqslant K\|f\|, \tag{4}$$

and

$$\left\|\sum_{i=m}^\infty f(x_i)f_i\right\|=\sup_{x\in B_X}\left|\sum_{i=m}^\infty f(x_i)f_i(x)\right|=\sup_{x\in B_X} f\left(\sum_{i=m}^\infty f_i(x)x_i\right). \tag{5}$$

(d) If (x_i, f_i) is an unconditional frame it follows from the Uniform Boundedness Principle that

$$K_u=\sup_{x\in B_X}\sup_{(\sigma_i)\subset\{\pm1\}}\left\|\sum\sigma_i f_i(x)x_i\right\|<\infty.$$

We call K_u the *unconditional constant of* (x_i, f_i).

The following proposition is a slight variation of [5, Theorem 2.6].

Proposition 2.4. *Let X be a separable Banach space and let $(x_i)_{i\in\mathbb{J}}\subset X$ and $(f_i)_{i\in\mathbb{J}}\subset X^*$, with $\mathbb{J}=\mathbb{N}$ or $\mathbb{J}=\{1,2,\ldots,N\}$ for some $N\in\mathbb{N}$.*

(a) *$(x_i, f_i)_{i\in\mathbb{J}}$ is a Schauder frame of X if and only if there is a Banach space Z with a Schauder basis $(z_i)_{i\in\mathbb{J}}$ and corresponding coordinate functionals (z_i^*), an isomorphic embedding $T: X\to Z$ and a bounded linear surjective map $S: Z\to X$, so that $S\circ T=\mathrm{Id}_X$ (i.e. X is isomorphic to a complemented subspace of Z), and $S(z_i)=x_i$, for $i\in\mathbb{J}$, and $T^*(z_i^*)=f_i$, for $i\in\mathbb{J}$, with $x_i\neq 0$. Moreover S and T can be chosen so that $\|S\|=1$ and $\|T\|\leqslant K$, where K is the projection constant of (x_i, f_i), and (z_i) can be chosen to be a bimonotone basis with $\|z_i\|=\|x_i\|$ if $i\in\mathbb{J}$, with $x_i\neq 0$.*

(b) *$(x_i, f_i)_{i\in\mathbb{J}}$ is an unconditional frame of X if and only if there is a Banach space Z with an unconditional basis (z_i) and corresponding coordinate functionals (z_i^*), an isomorphic embedding $T: X\to Z$ and a surjection $S: Z\to X$, so that $S\circ T=\mathrm{Id}_X$, $S(z_i)=x_i$, for $i\in\mathbb{J}$, and $T^*(z_i^*)=f_i$ for $i\in\mathbb{J}$, with $x_i\neq 0$.*

Proof. (a) Part "⇒." Assume that $(x_i, f_i)_{i\in\mathbb{J}}$ is a frame of X and let K be the projection constant of $(x_i, f_i)_{i\in\mathbb{J}}$. We put $\tilde{\mathbb{J}} = \{i \in \mathbb{J}\colon\ x_i \neq 0\}$, denote the unit vector basis of $c_{00}(\mathbb{J})$ by (z_i) and define on $c_{00}(\mathbb{J})$ the following norm $\|\cdot\|_Z$:

$$\Big\|\sum_{i\in\mathbb{J}} a_i z_i\Big\|_Z = \max_{m\leqslant n}\Big\|\sum_{i\in\tilde{\mathbb{J}}\cap\{m,m+1,\dots,n\}} a_i x_i\Big\|_X + \Big(\sum_{i\in\mathbb{J}\setminus\tilde{\mathbb{J}}} a_i^2\Big)^{1/2} \quad \text{for } (a_i)\subset\mathbb{R}. \tag{6}$$

It follows easily that (z_i) is a bimonotone basic sequence and, thus, a basis of the completion of $c_{00}(\mathbb{J})$ with respect to $\|\cdot\|_Z$, which we denote by Z.

The map

$$S : Z \to X, \qquad \sum a_j z_j \mapsto \sum a_j x_j,$$

is linear and bounded with $\|S\| = 1$. Secondly, define

$$T : X \to Z, \qquad x = \sum_{i\in\mathbb{J}} f_i(x)x_i = \sum_{i\in\tilde{\mathbb{J}}} f_i(x)x_i \mapsto \sum_{i\in\tilde{\mathbb{J}}} f_i(x)z_i.$$

Remark 2.3 (b) yields for $x \in X$

$$\Big\|\sum_{i\in\tilde{\mathbb{J}}} f_i(x)z_i\Big\|_Z = \sup_{m\leqslant n}\Big\|\sum_{i\in\tilde{\mathbb{J}}\cap\{m,m+1,\dots,n\}} f_i(x)x_i\Big\|_X = \sup_{m\leqslant n}\Big\|\sum_{i\in\mathbb{J}\cap\{m,m+1,\dots,n\}} f_i(x)x_i\Big\|_X \leqslant K\|x\|,$$

and, thus, that T is linear and bounded with $\|T\| \leqslant K$. Clearly it follows that $S \circ T = \mathrm{Id}_X$, which implies that T is an isomorphic embedding and that S is a surjection. Finally, if (z_i^*) are the coordinate functionals of (z_i) we deduce for $x \in X$ and $i \in \mathbb{J}$ that

$$T^*\big(z_i^*\big)(x) = z_i^*\big(T(x)\big) = \begin{cases} f_i(x) & \text{if } x_i \neq 0,\\ 0 & \text{if } x_i = 0,\end{cases}$$

which finishes the proof of " ⇒."

In order to show the converse in (a), assume that Z is a space with a basis $(z_i)_{i\in\mathbb{J}}$ and that $S : Z \to X$ is a bounded linear surjection, and $T : X \to Z$ an isomorphic embedding, with $S \circ T = \mathrm{Id}_X$. Put $x_i = S(z_i)$ and $f_i = T^*(z_i^*)$, for $i \in \mathbb{J}$. Then for $x \in X$,

$$x = S\circ T(x) = S\Big(\sum z_i^*\big(T(x)\big)z_i\Big) = \sum T^*\big(z_i^*\big)(x)S(z_i) = \sum f_i(x)x_i,$$

which implies that $(x_i, f_i)_{i\in\mathbb{J}}$ is a frame of X.

For the proof of (b) we replace (6) by

$$\Big\|\sum_{i\in\mathbb{J}} a_i z_i\Big\|_Z = \max_{(\sigma_i)\subset\{\pm1\}}\Big\|\sum_{i\in\tilde{\mathbb{J}}} \sigma_i a_i x_i\Big\|_X + \Big(\sum_{i\in\mathbb{J}\setminus\tilde{\mathbb{J}}} a_i^2\Big)^{1/2} \tag{7}$$

and note that arguments similar to those in the proof of (a) yield (b). □

Definition 2.5. Let (x_i, f_i) be a frame of a Banach space X and let Z be a space with a basis (z_i) and corresponding coordinate functionals (z_i^*). We call $(Z, (z_i))$ an *associated space to* (x_i, f_i) or a *sequence space associated to* (x_i, f_i) and (z_i) an *associated basis*, if

$$S : Z \to X, \qquad \sum a_i z_i \mapsto \sum a_i x_i \quad \text{and} \quad T : X \to Z, \qquad x = \sum f_i(x)x_i \mapsto \sum_{x_i\neq 0} f_i(x)z_i$$

are bounded operators. We call S the *associated reconstruction operator* and T the *associated decomposition operator* or *analysis operator*.

In this case, following [14] we call the triple $((x_i), (f_i), Z)$ an *atomic decomposition* of X.

Remark 2.6. By Proposition 2.4 the property of Banach space X to admit a frame is equivalent to the property of X being isomorphic to a complemented subspace of a space Z with basis. It was shown independently by Pełczyński [26] and Johnson, Rosenthal and Zippin [19] (see also [3, Theorem 3.13]) that the latter property is equivalent to X having the *Bounded Approximation Property*. X is said to have the *Bounded Approximation Property* if there is a $\lambda \geqslant 1$, so that for every $\varepsilon > 0$ and every compact set $K \subset X$ there is a finite rank operator $T : X \to X$ with $\|T\| \leqslant \lambda$ so that $\|T(x) - x\| \leqslant \varepsilon$ whenever $x \in K$.

Remark 2.7. Let $(x_j)_{j\in\mathbb{J}}$ be a Hilbert frame of a Hilbert space H and let Θ and I be defined as in the paragraph following Definition 2.1. We choose Z to be $\ell_2(\mathbb{J})$, $S=\Theta^*$ and

$$T=\Theta\circ I^{-1}: H\to\ell_2,\qquad x\mapsto\sum_{j\in\mathbb{J}}\langle I^{-1}(x),x_j\rangle e_j=\sum_{j\in\mathbb{J}}\langle x,I^{-1}(x_j)\rangle e_j$$

and observe that $S\circ T=\mathrm{Id}_H$, and for $j\in\mathbb{J}$ it follows that $S(e_j)=\Theta^*(e_j)=x_j$, and

$$T^*(e_j)(x)=\langle x,I^{-1}(x_j)\rangle\quad(x\in H)\quad\text{and, thus}\quad T^*(e_j)=I^{-1}(x_j).$$

Thus, if (x_i) is a Hilbert frame, then $((x_i),(I^{-1}(x_i))$ is a Schauder frame for which $Z=\ell_2(\mathbb{J})$ together with its unit vector basis is an associated space.

Conversely, let (x_i,f_i) be a Schauder frame of a Hilbert space H and assume that $Z=\ell_2(\mathbb{J})$ with its unit vector basis is an associated space. Denote by $T:H\to\ell_2(\mathbb{J})$ and $S:\ell_2(\mathbb{J})\to H$ the associated decomposition, respectively reconstruction operator. Then it follows that for all $x\in H$

$$\sum\langle x_i,x\rangle^2=\sum\langle S(e_i),x\rangle^2=\sum\langle e_i,S^*(x)\rangle^2=\|S^*(x)\|^2.$$

Thus, since S^* is an isomorphic embedding of H into ℓ_2, it follows that (x_i) is a Hilbert frame.

In the following observation we show that we can always *expand a frame by a bounded linear operator*.

Proposition 2.8. *Let (x_i,f_i) be a frame of a Banach space X and let Z be a space with a basis (z_i) which is associated to (x_i,f_i). Furthermore assume that Y is another space with a basis (y_i) and let $V:Y\to X$ be linear and bounded.*

Let $\mathbb{Z}t=Z\oplus_\infty Y$ (i.e. the product space $Z\times Y$ with the norm defined by $\|(z,y)\|_\infty=\max(\|z\|,\|y\|)$, for $z\in Z$ and $y\in Y$) and define $(\tilde z_i)\subset\tilde Z$, $(\tilde x_i)\subset X$ and $(\tilde f_i)\subset X^*$ by

$$\tilde z_i=\begin{cases}(z_{i/2},\lambda_{i/2}y_{i/2}),\\ (z_{(i+1)/2},-\lambda_{(i+1)/2}y_{(i+1)/2}),\end{cases}\qquad \tilde x_i=\begin{cases}x_{i/2}+\lambda_{i/2}V(y_{i/2}) & \text{if } i \text{ even},\\ x_{(i+1)/2}-\lambda_{(i+1)/2}V(y_{(i+1)/2}) & \text{if } i \text{ odd},\end{cases}$$

$$\tilde f_i=\begin{cases}\frac12 f_{i/2} & \text{if } i \text{ even},\\ \frac12 f_{(i+1)/2} & \text{if } i \text{ odd},\end{cases}$$

where $\lambda_j=\|z_j\|/\|y_j\|$, for $j\in\mathbb{N}$ (for example $\tilde z_1=z_1-\frac{\|z_1\|}{\|y_1\|}y_1$ and $\tilde z_2=z_1+\frac{\|z_1\|}{\|y_1\|}y_1$). Then $(\tilde x_i,\tilde f_i)$ is a frame of X, $(\tilde z_i)$ is a basis for $\tilde Z$ and $(\tilde Z,(\tilde z_i))$ is an associated space for $(\tilde x_i,\tilde f_i)$.

Proof. Let $T:X\to Z$ and $S:Z\to X$ be the associated decomposition and reconstruction operator, respectively. Note that the operators

$$\tilde S: Z\oplus_\infty Y\to X,\qquad (z,y)\mapsto S(z)+V(y)\quad\text{and}\quad \tilde T: X\to Z\oplus_\infty Y,\qquad x\mapsto\big(T(x),0\big)$$

are bounded and linear and that $\tilde S\circ\tilde T=\mathrm{Id}_X$ and $\tilde S(\tilde z_i)=\tilde x_i$, for $i\in\mathbb{N}$. It is easy to verify that $(\tilde z_i)$ is a basis of $\tilde Z$ for which its coordinate functionals $(\tilde z_i^*)$ are given by (denote the coordinate functionals of (y_i) by (y_i^*))

$$\tilde z_i^*=\begin{cases}\frac12(z^*_{i/2},\frac{1}{\lambda_i}y^*_{i/2}) & \text{if } i \text{ even},\\ \frac12(z^*_{(i+1)/2},-\frac{1}{\lambda_i}y^*_{(i+1)/2}) & \text{if } i \text{ odd}.\end{cases}$$

It follows for $x\in X$ that

$$\tilde T^*(\tilde z_i^*)(x)=\tilde z_i^*(\tilde T(x))=\left\{\begin{matrix} f_{i/2}(x)/2 & \text{if } i \text{ even},\\ f_{(i+1)/2}(x)/2 & \text{if } i \text{ odd}\end{matrix}\right\}=\tilde f_i(x),$$

which yields $\tilde T^*(\tilde z_i^*)=\tilde f_i$, for $i\in\mathbb{N}$. Thus, the claim follows from Proposition 2.4. □

3. Three examples

In [8] the following notion of quantization was introduced and studied for non-redundant systems.

Definition 3.1. Let $(x_i)_{i\in\mathbb{J}}$ be a fundamental system for X, with $\mathbb{J}=\mathbb{N}$ or $\mathbb{J}=\{1,2,\dots,N\}$, for some $N\in\mathbb{N}$ (i.e. $\overline{\mathrm{span}(x_i:\ i\in\mathbb{J})}=X$) and let $\varepsilon>0$ and $\delta>0$ be given. We say that $(x_i)_{i\in\mathbb{J}}$ has the (ε,δ)-*Net Quantization Property* (abbr. (ε,δ)-NQP) if for any $x=\sum_{i\in E}a_ix_i\in X$, $E\subset\mathbb{J}$ is finite, there exists a sequence $(k_i)\subset\mathbb{Z}$, $\mathrm{supp}(k_i)=\{i\in\mathbb{N}:\ k_i\neq0\}$ is finite, such that

$$\Big\|x-\sum k_i\delta x_i\Big\|\leqslant\varepsilon. \tag{8}$$

We say that (x_i) has the NQP if (x_i) has the (ε,δ)-NQP for some $\varepsilon>0$ and $\delta>0$.

When we ask whether or not in a certain representation of vectors the coefficients can be replaced by quantized coefficients, we are often interested in memorizing data as economically as possible, and reconstructing them with as little error as possible. With this in mind, we will exhibit in this section three examples, all of which satisfy the conditions (NQP) if we extend this notion word for word to frames. However, they show that it is not always meaningful to apply these notion to redundant systems like frames. These examples will then also guide us to more appropriate quantization concepts for frames in the next sections.

The first example (Example 3.2) is a *tight* Hilbert frame (f_i) in S_{ℓ_2} (i.e. $a = b$) consisting of normalized vectors so that for every $x \in \ell_2$ there is a sequence $(k_i) \subset \mathbb{Z}$ so that $\|x - \sum_{i\in\mathbb{N}} k_i f_i\| < 1$. But these coefficients (k_i) might get arbitrarily large for elements in B_{ℓ_2}. This means that we would need an infinite alphabet to approximate vectors which are in B_{ℓ_2}.

The second example (Example 3.3) is a semi-normalized Hilbert frame (f_i) in ℓ_2 which has the property that for every $x \in \ell_2$ there is a sequence $(k_i) \subset \mathbb{Z}$, so that $\|x - \sum_{i\in\mathbb{N}} k_i f_i\| < 1$ and (k_i) has the additional property that $\max_{i\in\mathbb{N}} |k_i| \leqslant 1$, if $x \in B_{\ell_2}$.

The third example (Example 3.4) is a Schauder frame (f_i) of ℓ_2 which has the property that for every $x \in \ell_2$ there is a sequence $(k_i) \subset \mathbb{Z}$, so that not only $\max_{i\in\mathbb{N}} |k_i| \leqslant \|x\|$ and $\|x - \sum_{i\in\mathbb{N}} k_i f_i\| < 1$, but so that also the support of (k_i), i.e. the set $\{i \in \mathbb{N}\colon k_i \neq 0\}$ is uniformly bounded.

However, in order to approximate even the vectors of a given finite dimensional subspace in Examples 3.3 and 3.4, we need to use an overproportional number of elements of the frame. More precisely, in both examples the number of frame elements we would need to approximate the elements of a finite dimensional subspace is not linear with respect to the dimension.

Finally note that in all three cases we used the fact that a vector can be represented in more than one way by the elements of the frame, unlike in the situation of bases.

Example 3.2. Let $0 < \varepsilon_i < 1/2$. For $i \in \mathbb{N}$ define the following vectors f_{2i-1} and f_{2i} in S_{ℓ_2}.

$$f_{2i-1} = \sqrt{1-\varepsilon_i^2}\, e_{2i-1} + \varepsilon_i e_{2i} \quad \text{and} \quad f_{2i} = -\varepsilon_i e_{2i-1} + \sqrt{1-\varepsilon_i^2}\, e_{2i}.$$

Clearly (f_i) is an orthonormal basis for ℓ_2 and we let $\mathcal{F} = \{e_i\colon i \in \mathbb{N}\} \cup \{f_i\colon i \in \mathbb{N}\}$. Then $\mathcal{F}$ is a tight frame (as is any finite union of orthonormal bases) and the sequence (z_i) with

$$z_{2i-1} = e_{2i-1} - f_{2i-1} = \bigl(1 - \sqrt{1-\varepsilon_i^2}\bigr) e_{2i-1} - \varepsilon_i e_{2i},$$
$$z_{2i} = e_{2i} - f_{2i} = \varepsilon_i e_{2i-1} + \bigl(1 - \sqrt{1-\varepsilon_i^2}\bigr) e_{2i}, \quad \text{for } i \in \mathbb{N},$$

is an orthogonal basis and $\|z_{2i-1}\| = \|z_{2i}\| = O(\varepsilon_i)$. Thus, if (ε_i) converges fast enough to 0 it follows that for any $x \in \ell_2$ there is a family $(k_i) \subset \mathbb{Z}$, with $|\operatorname{supp}(k_i)| < \infty$, so that

$$\Bigl\| x - \sum k_i z_i \Bigr\| = \Bigl\| x - \sum k_i (e_i - f_i) \Bigr\| < 1.$$

Example 3.3. Our second example is a semi-normalized Hilbert frame (x_i) in ℓ_2 so that

$$D = \left\{ \sum_{i=1}^{\infty} k_i x_i\colon\ k_i \in \{-1, 0, 1\} \text{ and } \{i\colon k_i \neq 0\} \text{ is finite} \right\}$$

is dense in B_{ℓ_2}.

Put $(c_i)_{i=1}^{\infty} = (1/2^i)$ and partition the unit vector basis (e_i) of ℓ_2 into infinitely many subsequences of infinite length, say $(e(i,j)\colon i, j \in \mathbb{N})$. Then our frame (f_k) is defined to be the sequence:

$$f_1 = c_1 e_1 + e(1,1), \quad f_2 = e(1,1),$$
$$f_3 = c_1 e_2 + e(1,2), \quad f_4 = e(1,2), \quad f_5 = c_2 e_1 + e(2,1), \quad f_6 = e(2,1),$$
$$f_7 = c_1 e_3 + e(1,3), \quad f_8 = e(1,3), \quad f_9 = c_2 e_2 + e(2,2),$$
$$f_{10} = e(2,2), \quad f_{11} = c_3 e_1 + e(3,1), \quad f_{13} = e(3,1),$$
$$f_{14} = c_1 e_4 + e(1,4), \quad f_{15} = e(1,4), \quad \ldots, \quad f_{20} = c_4 e_1 + e(4,1), \quad f_{21} = e(4,1),$$
$$\vdots$$

Note that the set of vectors $x \in B_{\ell_2}$ of the form

$$x = \sum_{i,j\in\mathbb{N}} \varepsilon(i,j) c_i e_j = \sum_{i,j\in\mathbb{N}} \bigl(\varepsilon(i,j)\bigl(c_i e_j + e(i,j)\bigr) - \varepsilon(i,j) e(i,j)\bigr)$$

where $(\varepsilon(i,j)) \subset \{-1,0,1\}$ so that the set $\{i,j \in \mathbb{N}:\ \varepsilon(i,j) \neq 0\}$ is finite, are dense in B_{ℓ_2}. This implies that every $x \in B_{\ell_2}$ is the limit of vectors (x_n) with

$$(x_n) \subset \Big\{ \sum \varepsilon_i f_i:\ (\varepsilon_i) \subset \{-1,0,1\},\ \{i \in \mathbb{N}:\ \varepsilon_i \neq 0\} \text{ is finite} \Big\}.$$

The sequence (f_n) is a frame. Indeed for any $x = \sum x_i e_i \in \ell_2$ we have

$$\sum \langle f_i, x\rangle^2 = \sum_{i \text{ even}} \langle f_i, x\rangle^2 + \sum_{i \text{ odd}} \langle f_i, x\rangle^2 = \sum x_i^2 + \sum_{i,j\in\mathbb{N}} \big(c_i x_j + \langle e(i,j), x\rangle\big)^2$$
$$\begin{cases} \geqslant \|x\|^2, \\ \leqslant \sum x_i^2 + \sum_{i,j\in\mathbb{N}} 2c_i^2 x_j^2 + 2\langle e(i,j), x\rangle^2 \leqslant \Big(3 + 2\sum c_i^2\Big)\|x\|^2. \end{cases}$$

Example 3.4. We construct a Schauder frame (x_i, f_i) of ℓ_2 so that (x_n) is dense in B_{ℓ_2}.

Let (y_n) be dense in B_{ℓ_2} and choose for each $n \in \mathbb{N}$

$$x_{2n-1} = y_n + e_n, \qquad x_{2n} = y_n, \qquad f_{2n-1} = e_n \quad \text{and} \quad f_{2n} = -e_n.$$

Clearly, for every $x \in \ell_2$

$$x = \sum \langle e_i, x\rangle e_i = \sum \big(\langle f_{2n-1}, x\rangle x_{2n-1} + \langle f_{2n}, x\rangle x_{2n}\big)$$

(the above sum is conditionally converging). It follows that $((x_n),(f_n))$ is a Schauder frame of ℓ_2. It is clear that (x_n) is not a Hilbert frame.

4. Quantization with Z-bounded coefficients

One way one might avoid examples like the ones mentioned in Section 3 is to impose boundedness conditions on the quantized coefficients within an associated space Z.

Definition 4.1. Assume $(x_i, f_i)_{i\in\mathbb{J}}$, $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1,2,\ldots,N\}$, for some $N \in \mathbb{N}$, is a frame of a Banach space X. Let Z be a space with basis (z_i) which is associated to $(x_i, f_i)_{i\in\mathbb{J}}$. Let $\varepsilon, \delta > 0$, $C \geqslant 1$.

We say that (x_i, f_i) *satisfies the* (ε,δ,C)*-Net Quantization Property with respect to* $(Z,(z_i))$ or (ε,δ,C)–Z-NQP, if for all $x \in X$ there exists a sequence $(k_i)_{i\in\mathbb{J}} \subset \mathbb{Z}$ with finite support so that

$$\Big\| \sum k_i \delta z_i \Big\|_Z \leqslant C\|x\| \quad \text{and} \quad \Big\| x - \sum \delta k_i x_i \Big\|_X \leqslant \varepsilon. \tag{9}$$

We say that (x_i, f_i) *satisfies the NQP with respect to* $(Z,(z_i))$ if it satisfies the (ε,δ,C)–Z-NQP for some choice of $\varepsilon, \delta > 0$ and $C \geqslant 1$.

It is easy to see that the property (ε,δ,C)-NQP with respect to some associated space is homogeneous in (ε,δ), meaning that a frame (x_i, f_i) is (ε,δ,C)-NQP if and only if for some $\lambda > 0$ (or for all λ) (x_i, f_i) satisfies the $(\lambda\varepsilon, \lambda\delta, C)$-NQP. The following result, analogous to [8, Theorem 2.4], shows that it is enough to verify that one can quantize the coefficients of elements x which are in B_X to deduce the NQP.

Proposition 4.2. *Assume that (x_i) and (z_i) are some sequences in Banach spaces X and Z, respectively, and assume that there are $C_0 < \infty$, $\delta_0 > 0$ and $0 < q_0 < 1$, so that for all $x \in B_X$ there is a sequence $(k_i) \subset \mathbb{Z}$, $(k_i) \in c_{00}$ with*

$$\Big\| \sum \delta_0 k_i z_i \Big\| \leqslant C_0 \quad \textit{and} \quad \Big\| x - \sum \delta_0 k_i x_i \Big\| \leqslant q_0. \tag{10}$$

Then there are $\delta_1 > 0$, and $C_1 < \infty$ only depending on δ_0, q_0 and C_0 so that for all $x \in X$ there is a sequence $(k_i) \subset \mathbb{Z}$, $(k_i) \in c_{00}$, with

$$\Big\| \sum \delta_1 k_i z_i \Big\| \leqslant C_1\|x\| \textit{and} \quad \Big\| x - \sum \delta_1 k_i x_i \Big\| \leqslant 1. \tag{11}$$

Proof. Choose $n_1 \in \mathbb{N}$ and q_1 so that

$$\frac{n_1+1}{n_1} q_0 = q_1 < 1 \tag{12}$$

and put $\delta_1 = \delta_0/n_1$.

We first claim that for any $0 < \delta \leqslant \delta_1$ and any $x \in B_X$ there is a sequence $(k_i) \in \mathbb{Z}^{\mathbb{N}} \cap c_{00}$ so that

$$\Big\| \sum k_i \delta z_i \Big\| \leqslant 2C_0 \text{and} \quad \Big\| x - \sum \delta k_i x_i \Big\| \leqslant q_1. \tag{13}$$

Indeed, let $\delta \leqslant \delta_1$ and $x \in B_X$ and choose $n \geqslant n_1$ in $\mathbb{N}$ so that $\frac{\delta_0}{n+1} < \delta \leqslant \frac{\delta_0}{n}$ and $(k_i) \subset \mathbb{Z}$ so that

$$\Bigl\|\sum k_i\delta_0 z_i\Bigr\| \leqslant C_0 \quad \text{and} \quad \Bigl\|\frac{\delta_0}{\delta(n+1)}x - \sum k_i\delta_0 x_i\Bigr\| \leqslant q_0$$

and, thus, since $n \geqslant n_1$,

$$\Bigl\|\sum k_i(n+1)\delta z_i\Bigr\| = \Bigl\|\sum k_i\delta_0 z_i\Bigr\|\frac{\delta(n+1)}{\delta_0} \leqslant \frac{n+1}{n}C_0 \leqslant 2C_0$$

and

$$\Bigl\|x - \sum k_i\delta(n+1)x_i\Bigr\| \leqslant \Bigl\|\frac{\delta_0}{(n+1)\delta}x - \sum k_i\delta_0 x_i\Bigr\|\frac{\delta(n+1)}{\delta_0} \leqslant q_0\frac{\delta}{\delta_0}(n+1) \leqslant q_1.$$

By induction on $n \in \mathbb{N}$ we show that for any $\delta \leqslant q_1^{n-1}\delta_1$ and any $x \in B_X$ there is a $(k_i) \subset \mathbb{Z}$, $(k_i) \in c_{00}$, so that

$$\Bigl\|\sum k_i\delta z_i\Bigr\| \leqslant 2C_0\sum_{i=0}^{n-1}q_1^i \quad \text{and} \quad \Bigl\|x - \sum k_i\delta x_i\Bigr\| \leqslant q_1^n. \tag{14}$$

For $n = 1$ this is just (13). Assume our claim to be true for n and let $\delta \leqslant \delta_1 q_1^n$ and $x \in B_X$. By our induction hypothesis, we can find $(k_i) \subset \mathbb{Z}$, $(k_i) \in c_{00}$, so that (14) holds. Since $q_1^{-n}(x - \sum k_i\delta x_i) \in B_X$ and since $\delta q_1^{-n} \leqslant \delta_1$, we can use our first claim and choose $(\tilde{k}_i) \in \mathbb{Z}^{\mathbb{N}} \cap c_{00}$ so that

$$\Bigl\|\sum \tilde{k}_i\delta q_1^{-n}z_i\Bigr\| \leqslant 2C_0 \quad \text{and} \quad \Bigl\|q_1^{-n}\Bigl(x - \sum k_i\delta x_i\Bigr) - \sum \delta q_1^{-n}\tilde{k}_i x_i\Bigr\| \leqslant q_1,$$

and, thus,

$$\Bigl\|\sum (k_i + \tilde{k}_i)\delta z_i\Bigr\| \leqslant 2C_0\sum_{i=0}^{n-1}q_1^i + 2C_0 q_1^n,$$

and

$$\Bigl\|x - \sum \delta k_i x_i - \sum \delta\tilde{k}_i x_i\Bigr\| \leqslant q_1^{n+1},$$

which finishes the induction step.

Now define $C_1 = 2C_0\sum_{n=0}^{\infty}q_1^n$ and let $x \in X$ be arbitrary.

If $\|x\| \geqslant 1$ (this is the only case left to consider) we choose $n \in \mathbb{N}$ with $q_1^n < \frac{1}{\|x\|} \leqslant q_1^{n-1}$ and, by (14) we can choose $(k_i) \in \mathbb{Z}^{\mathbb{N}} \cap c_{00}$ so that

$$\Bigl\|\sum k_i\frac{\delta_1}{\|x\|}z_i\Bigr\| \leqslant 2C_0\sum_{i=0}^{n-1}q_1^i \leqslant C_1 \quad \text{and} \quad \Bigl\|\frac{x}{\|x\|} - \sum\frac{k_i\delta_1}{\|x\|}x_i\Bigr\| \leqslant q_1^n,$$

which yields

$$\Bigl\|\sum k_i\delta_1 z_i\Bigr\| \leqslant C_1 q_1\|x\| \quad \text{and} \quad \Bigl\|x - \sum k_i\delta_1 x_i\Bigr\| \leqslant q_1^n\|x\| \leqslant 1. \qquad \square$$

In the following result we consider a finite frame $(x_i, f_i)_{i=1}^N$ of a finite dimensional Banach space X, and exploit the fact that, if $(x_i, f_i)_{i=1}^N$ has the (ε, δ, C)-NQP with respect to some space Z having a basis $(z_i)_{i=1}^N$, then the value

$$\varepsilon^{-n} = \mathrm{Vol}(B_X)/\mathrm{Vol}(\varepsilon B_X)$$

must be smaller then the cardinality of the set

$$\mathcal{F}_{(\delta,C)}(x_i) = \Bigl\{\sum n_j\delta x_j\colon \Bigl\|\sum n_j\delta z_j\Bigr\| \leqslant C\Bigr\}.$$

Proposition 4.3. *Assume $f : \mathbb{N} \to \mathbb{R}^+$, $f(1) = 1$, is strictly increasing to ∞, $C \geqslant 1$, and $0 \leqslant \delta, \varepsilon < 1$.*

Let $(x_i, f_i)_{i=1}^N$ be a frame of a Banach space X with $n := \dim(X) < \infty$ and let Z be an N-dimensional space, $N \in \mathbb{N}$, with a normalized basis $(z_i)_{i=1}^N$, which is associated to $(x_i, f_i)_{i=1}^N$. Let $S : Z \to X$ be the reconstruction operator, and denote by K the projection constant of (z_i). Assume ($\#A$ denotes the cardinality of a set $A \subset \mathbb{N}$)

$$\mathcal{F}_{(\delta,C)}(x_i) \text{ is } \varepsilon\text{-dense in } B_X, \quad \text{and} \tag{15}$$

$$f(\#A) \leqslant \Bigl\|\sum_{j\in A}n_j z_j\Bigr\|, \quad \text{whenever } A \subset \{1, 2, \ldots, N\} \text{ and } (n_j)_{j\in A} \subset \mathbb{Z}\setminus\{0\}. \tag{16}$$

Then

$$\ln N \geqslant \frac{n\ln(1/\varepsilon)}{f^{-1}(C/\delta)} - \ln\left(\frac{2KC}{\delta}+1\right). \tag{17}$$

Moreover, assume that $g:\mathbb{N}\to\mathbb{R}$ *is strictly increasing and satisfies*

$$\lim_{n\to\infty}\frac{g(n)\ln n}{n}=0, \tag{18}$$

and let X, Z, K, $(x_i,f_i)_{i=1}^N$, $(z_i)_{i=1}^N$, *and* S *be as above satisfying* (15), (16) *and, moreover,*

$$g(\#A)\geqslant \sup\left\|\sum_{j\in A}\pm z_j\right\|, \quad \text{whenever } A\subset\{1,2,\dots,N\}, \tag{19}$$

Then it follows that $\dim(X)\leqslant n_0$, *where* n_0 *depends only on* K, C, $\|S\|$, ε, δ, f *and* g *and is increasing in* K, C, $\|S\|$.

Before providing a proof let us explain Proposition 4.3 in more detail.

Remark 4.4. Condition (16) means that we are in a certain sense "far" away from the space ℓ_∞^n (in which quantization can be achieved trivially). Condition (19) implies that the associated basis is "far" from the ℓ_1-basis (which also be used to quantize, as Proposition 4.7 shows). To be more concrete, we consider the following situation.

Assume, for example, that (z_i) has an ℓ_p lower bound, for some $1\leqslant p<\infty$, i.e. that there is a $D_1<\infty$ so that

$$\left(\sum_{i=1}^N|a_i|^p\right)^{1/p}\leqslant D_1\left\|\sum_{i=1}^N a_iz_i\right\| \quad \text{for } (a_i)\subset\mathbb{R}.$$

This means that condition (16) holds if we let $f(m)=m^{1/p}/D_1$. Also assume for simplicity that $K=1$.

Proposition 4.3 now implies that if X is a finite dimensional Banach space X admitting a frame of length N whose associated space is Z and which satisfies the quantization condition (15) for some $C\geqslant1$, ε and $\delta>0$ then N is at least an exponential function of the dimension of X.

If we, moreover, require that (z_i) has also an upper ℓ_q-estimate, for some $1<q$, i.e. if for some D_2

$$\left\|\sum_{i=1}^N a_iz_i\right\|\leqslant D_2\left(\sum_{i=1}^N|a_i|^q\right)^{1/q} \quad \text{for } (a_i)\subset\mathbb{R},$$

then the second part of Proposition 4.3 yields that the dimension of X has to be bounded by a constant only depending on p, q, D_1 D_2, C, ε and δ.

Proof of Proposition 4.3. First note that, if $A\subset\{1,2,\dots,N\}$ and $(n_j)_{j\in A}\subset\mathbb{Z}\setminus\{0\}$ with $C\geqslant\|\sum_{j\in A}n_j\delta z_j\|_Z\geqslant\delta f(\#A)$, then $\#A\leqslant f^{-1}(C/\delta)$ and $|n_j|\leqslant KC/\delta$ for $j\in A$. Thus, (15) and the volume argument, mentioned before the statement of our proposition, yields

$$\varepsilon^{-n}\leqslant\#\mathcal{F}_{(\delta,C)}(x_i)\leqslant\binom{N}{\lfloor f^{-1}(C/\delta)\rfloor}\left(\frac{2KC}{\delta}+1\right)^{\lfloor f^{-1}(C/\delta)\rfloor}\leqslant\left[N\left(\frac{2KC}{\delta}+1\right)\right]^{f^{-1}(C/\delta)},$$

which, after taking $\ln(\cdot)$ on both sides, implies (17).

Now assume that also (19) is satisfied. Let $(e_i,e_i^*)_{i=1}^n$ be an Auerbach basis of X, i.e. $\|e_i\|=\|e_i^*\|=1$ and $e_i^*(e_j)=\delta_{(i,j)}$. Such a basis always exists (cf. [11, Theorem 5.6]). Choose $0<\eta<\infty$ so that $\varepsilon(1+1/\eta)<1$ and define for $i=1,2,\dots,n$,

$$A_i=\left\{j\in\{1,2,\dots,N\}:\ e_i^*(x_j)\geqslant\varepsilon^{-1}\eta KCf^{-1}(C/\delta)n\right\}.$$

Then it follows for the right choice of $\sigma_j=\pm1$, $j\in A_i$, that

$$\begin{aligned} g(\#A_i)&\geqslant\left\|\sum_{j\in A_i}\pm z_j\right\|\\ &\geqslant\frac{1}{\|S\|}\sup\left\|\sum_{j\in A_i}\pm x_j\right\|\\ &\geqslant\frac{1}{\|S\|}e_i^*\left(\sum_{j\in A_i}\sigma_jx_j\right)\geqslant\frac{\#A_i\varepsilon}{\eta\|S\|KCf^{-1}(C/\delta)n}\end{aligned}$$

and thus

$$\frac{\#A_i}{g(\#A_i)} \leqslant \frac{\eta\|S\|KCf^{-1}(C/\delta)n}{\varepsilon}. \tag{20}$$

Put $A = \bigcup_{i\leqslant n} A_i$. If $(n_j)_{j\leqslant N} \subset \mathbb{Z}$ is such that $\sum_{j=1}^N \delta n_j x_j \in \mathcal{F}_{(\delta,C)}(x_j)$, then

$$\left\| \sum_{j\in A^c} \delta n_j x_j \right\| \leqslant \sum_{j\in A^c,\, n_j\neq 0} \left(\sum_{i=1}^n \left|e_i^*(x_j)\right| \right) \max_{j\leqslant N} \delta|n_j| \leqslant \frac{\varepsilon\#\{j:\ n_j\neq 0\}}{\eta f^{-1}(C/\delta)} \leqslant \frac{\varepsilon}{\eta},$$

where the second inequality follows from the definition of the A_i's and the observation at the beginning of the proof, and the last inequality follows from the fact that

$$f\bigl(\#\{j:\ n_j\neq 0\}\bigr) \leqslant \left\| \sum n_j z_j \right\| \leqslant C/\delta. \tag{21}$$

This implies together with (15) that the set

$$\tilde{\mathcal{F}}_{(\delta,C)} = \left\{ \sum_{j\in A} n_j \delta x_j:\ (n_j)\subset \mathbb{Z},\ \left\| \sum_{j=1}^N n_j \delta z_j \right\| \leqslant C \right\}$$

is $\varepsilon(1+\frac{1}{\eta})$-dense in B_X. Hence, our usual argument comparing volumes and (21) yields

$$\#A^{f^{-1}(C/\delta)} \left(\frac{2KC}{\delta} + 1 \right)^{f^{-1}(C/\delta)} \geqslant \binom{\#A}{\lfloor f^{-1}(C/\delta)\rfloor} \left(\frac{2KC}{\delta} + 1 \right)^{\lfloor f^{-1}(C/\delta)\rfloor} \geqslant \frac{1}{\varepsilon^n(1+1/\eta)^n}.$$

Taking $\ln(\cdot)$ on both sides and letting $r(\ell) = \ln(\ell)g(\ell)/\ell$ for $\ell\in\mathbb{N}$, and since $\varepsilon(1+1/\eta) < 1$ we conclude by (20) that

$$\begin{aligned}
n\ln\left(\frac{1}{\varepsilon(1+1/\eta)}\right) &\leqslant f^{-1}(C/\delta)\left(\ln(\#A) + \ln\left(\frac{2KC}{\delta}+1\right)\right) \\
&\leqslant f^{-1}(C/\delta)\left(\ln n + \max_{i\leqslant n}\ln(\#A_i) + \ln\left(\frac{4KC}{\delta}\right)\right) \\
&= f^{-1}(C/\delta)\left(\ln n + \max_{i\leqslant n} r(\#A_i)\left(\frac{\#A_i}{g(\#A_i)}\right) + \ln\left(\frac{4KC}{\delta}\right)\right) \\
&\leqslant f^{-1}(C/\delta)\left(\ln n + nr(\#A_{i_0})\frac{\eta\|S\|KCf^{-1}(C/\delta)}{\varepsilon} + \ln\left(\frac{4KC}{\delta}\right)\right)
\end{aligned}$$

where $i_0 \leqslant n$ is chosen so that $\#A_{i_0}$ is maximal. By our assumption on g we can find an $\ell_0\in\mathbb{N}$ so that

$$\eta\|S\|KCf^{-1}(C/\delta)r(\ell) \leqslant \frac{1}{2}\ln\left(\frac{1}{\varepsilon(1+1/\eta)}\right), \quad \text{whenever } \ell \geqslant \ell_0.$$

If $\#A_{i_0} \leqslant \ell_0$ then

$$n\ln\left(\frac{1}{\varepsilon(1+1/\eta)}\right) \leqslant f^{-1}(C/\delta)\left[\ln n + \ln\ell_0 + \ln\left(\frac{2KC}{\delta}+1\right)\right],$$

which implies that n is bounded by a number which only depends on ε, δ, C, f, g and K. If $\#A_{i_0} > \ell_0$, then it follows that

$$\frac{n}{2}\ln\left(\frac{1}{\varepsilon(1+1/\eta)}\right) \leqslant f^{-1}(C/\delta)\left(\ln n + \ln\left(\frac{4KC}{\delta}\right)\right)$$

which implies our claim also in that case. □

We shall formulate a corollary of Proposition 4.3 for the infinite dimensional situation. We need first to introduce some notation and make some observations.

Let $(x_i, f_i)_{i\in\mathbb{N}}$ be a frame of X. Furthermore assume that X has the *π_λ-property*, which means that there is a sequence $\overline{P} = (P_n)$ of finite rank projections, whose norms are uniformly bounded, and which *approximate the identity*, i.e.

$$x = \lim_{n\to\infty} P_n(x), \quad \text{in norm for all } x\in X. \tag{22}$$

For example, if X has a basis (e_i) we could choose for $n\in\mathbb{N}$ the projection onto the first n coordinates, i.e.

$$P_n: X\to X, \qquad \sum a_i e_i \mapsto \sum_{i=1}^n a_i e_i.$$

It is easy to see that $(P_n(x_i), f_i|_{P_n(X)})$ is a frame of the space $X_n = P_n(X)$. Moreover, condition (22) and a straightforward compactness argument shows that for any $n \in \mathbb{N}$ and any $\frac{1}{2} < r < 1$ there is an $M_n = M(r,n)$ so that it follows that

$$\left\| x - \sum_{i=1}^{N} \langle f_i, x\rangle P_n(x_i) \right\| \leqslant (1-r)\|x\|, \quad \text{whenever } x \in X_n \text{ and } N \geqslant M_n. \tag{23}$$

It follows that the operators (Q_n), with

$$Q_n : X_n \to X_n, \qquad x \mapsto \sum_{i=1}^{M_n} \bigl\langle P_n^*(f_i), x\bigr\rangle P_n(x_i) = \sum_{i=1}^{M_n} \langle f_i, x\rangle P_n(x_i),$$

are uniformly bounded ($\|Q_n\| \leqslant 2$, for $n \in \mathbb{N}$), invertible and their inverses are uniformly bounded ($\|Q_n^{-1}\| \leqslant \frac{1}{r}$, for $n \in \mathbb{N}$). For $x \in X_n$ we write

$$x = Q_n^{-1} Q_n(x) = \sum_{i=1}^{M_n} \bigl\langle P_n^*(f_i), x\bigr\rangle \bigl(Q_n^{-1} \circ P_n\bigr)(x_i)$$

and deduce therefore that

$$\bigl(y_i^{(n)}, g_i^{(n)}\bigr)_{i=1}^{M_n} := \bigl(\bigl(Q_n^{-1} \circ P_n\bigr)(x_i), P_n^*(f_i)\bigr)_{i=1}^{M_n}$$

is a finite frame of X_n.

Let Z be a space with basis (z_i) which is associated to the frame (x_i, f_i). It follows easily that the operators (S_n) and (T_n)

$$S_n : Z_n = [z_i\colon\ i \leqslant M_n] \to X_n, \qquad z \mapsto \sum_{i=1}^{M_n} a_i z_i \mapsto \sum_{i=1}^{M_n} a_i \bigl(Q_n^{-1} \circ P_n\bigr)(x_i),$$

$$T_n : X_n \to Z_n, \qquad x = \sum_{i=1}^{M_n} f_i(x)\bigl(Q_n^{-1} \circ P_n\bigr)(x_i) \mapsto \sum_{i=1}^{M_n} f_i(x) z_i$$

are uniformly bounded, and thus Z_n is an associated space for the frame $(y_i^{(n)}, g_i^{(n)})_{i \leqslant M_n}$ while T_n and S_n are the associated decomposition and reconstruction operators, respectively.

Finally assume that the frame (x_i, f_i) satisfies the (ε, δ, C)-NQP with respect to Z. Again by compactness and using Proposition 4.2 we can choose $M_n = M(r,n)$ large enough so that it also satisfies the following:

For all $n \in \mathbb{N}$ and all $x \in X_n$ there is a sequence $(k_i)_{i=1}^{M_n} \subset \mathbb{Z}$ so that

$$\left\| \sum_{i=1}^{M_n} k_i \delta z_i \right\| \leqslant C\|x\| \quad \text{and} \quad \left\| x - \sum_{i=1}^{M_n} \delta k_i x_i \right\| \leqslant \varepsilon. \tag{24}$$

After changing $\varepsilon > 0$ and δ proportionally, if necessary, and since $r > \frac{1}{2}$, we can assume that $q = \frac{1-r}{r} + \sup_n \|P_n\| \frac{\varepsilon}{r} < 1$. For n in $\mathbb{N}$ and $x \in B_{X_n}$ we can therefore choose $(k_i)_{i=1}^{M_n} \subset \mathbb{Z}$ so that $\|\sum_{i=1}^{M_n} \delta k_i z_i\| \leqslant C$ and

$$\begin{aligned}
\left\| x - \sum_{i=1}^{M_n} \delta k_i \bigl(Q_n^{-1} \circ P_n\bigr)(x_i) \right\| &\leqslant \bigl\|Q_n^{-1}\bigr\| \cdot \left\| Q_n(x) - \sum_{i=1}^{M_n} \delta k_i P_n(x_i) \right\| \\
&\leqslant \bigl\|Q_n^{-1}\bigr\| \cdot \bigl\|Q_n(x) - x\bigr\| + \bigl\|Q_n^{-1}\bigr\| \cdot \left\| x - \sum_{i=1}^{M_n} \delta k_i P_n(x_i) \right\| \\
&\leqslant \bigl\|Q_n^{-1}\bigr\| \cdot \bigl\|Q_n(x) - x\bigr\| + \bigl\|Q_n^{-1}\bigr\| \cdot \|P_n\| \cdot \left\| x - \sum_{i=1}^{M_n} \delta k_i x_i \right\| \\
&\leqslant \frac{1-r}{r} + \sup_n \|P_n\| \frac{\varepsilon}{r} = q < 1.
\end{aligned}$$

Thus, for every $n \in \mathbb{N}$ the frame $(P_n^*(f_i), (Q_n^{-1} \circ P_n)(x_i))_{i=1}^{M_n}$ satisfies condition (15) of Proposition 4.3 (for $\varepsilon = q$). Therefore we deduce the following corollary.

Corollary 4.5. *Let $(x_i, f_i)_{i \in \mathbb{N}}$ be a frame of an infinite dimensional Banach space X for which there is a uniformly bounded sequence (P_n) of finite rank projections which approximate the identity. Assume that $(x_i, f_i)_{i \in \mathbb{N}}$ satisfies the (ε, δ, C)-NQP with respect to a space Z with basis (z_i) for some choice of $\varepsilon > 0$, $\delta > 0$ and C so that $q = \frac{1-r}{r} + \sup_n \|P_n\| \frac{\varepsilon}{r} < 1$ with $\frac{1}{2} < r < 1$. Let (M_n) be any sequence in $\mathbb{N}$ which satisfies* (23) *and* (24)*.*

Finally assume that (z_i) satisfies the following lower estimate

$$\lim_{n\to\infty}\inf\left\{\left\|\sum_{j\in A} n_j z_j\right\|:\ (n_j)_{j\in A}\subset\mathbb{Z}\setminus\{0\},\ A\subset\mathbb{N},\ \#A=n\right\}=\infty. \tag{25}$$

Then

(a) *(M_n) increases exponentially with the dimension of X_n, i.e. there is a $c>1$, so that $M_n \geqslant c^{\dim(X_n)}$ eventually,*

(b) $$\limsup_{n\to\infty}\frac{\ln(n)}{n}\sup\left\{\left\|\sum_{i\in A}\pm z_i\right\|:\ A\subset\mathbb{N},\ \#A=n\right\}>0.$$

Let us simplify the conditions in Corollary 4.5. Note that (z_i) satisfies (25) if it satisfies lower ℓ_q estimates for some $q<\infty$ (see Remark 2.6), but that it cannot satisfy the conclusion (b) of Corollary 4.5 in case it satisfies upper ℓ_p estimates for some $p>0$. We therefore deduce the following corollary.

Corollary 4.6. *Assume that X is an infinite dimensional Banach space with the π_λ-property and that Z is a Banach space with a basis (z_i) satisfying for some choice of $1<q<p<\infty$ lower ℓ_p and upper ℓ_q estimates.*

Then no frame of X has the NQP with respect to $(Z,(z_i))$.

The following example shows how to construct a frame with respect to a space Z which contains ℓ_1.

Proposition 4.7. *Let $(x_i,f_i)_{i\in\mathbb{N}}$ be any frame of a Banach space X and let Z be a space with semi-normalized basis (z_i), which is associated to (x_i,f_i). Then there is a frame $(\tilde{x}_i,\tilde{f}_i)_{i\in\mathbb{N}}$ and a basis $(\tilde{z}_i)$ of $\tilde{Z}=Z\oplus_\infty\ell_1$ so that $(\tilde{x}_i,\tilde{f}_i)_{i\in\mathbb{N}}$ has $\tilde{Z}$ as an associated space and has the NQP with respect to $\tilde{Z}$. Moreover, $(\tilde{x}_i,\tilde{f}_i)_{i\in\mathbb{N}}$ is semi-normalized if $(x_i,f_i)_{i\in\mathbb{N}}$ has this property (for example if (x_i) is a normalized basis of X).*

Proof. Assume, without loss of generality that $\|z_i\|=1$ for $i\in\mathbb{N}$. Choose a quotient map $Q:\ell_1\to X$ so that $(Q(e_i):\ i\in\mathbb{N})$ is a $\frac{1}{2}$-net in B_X and so that $\|Q(e_i)\pm x_i\|>\frac{1}{4}$ for $i\in\mathbb{N}$ (which is easy to accomplish). Finally we apply Proposition 2.8 to $Y=\ell_1$ with its unit vector basis (e_i) and $V=Q$, and observe that the frame $(\tilde{x}_i,\tilde{f}_i)$ and basis $(\tilde{z}_i)$ of $\tilde{Z}$, as constructed there, has the property that for any $x\in B_X$ there is an $i\in\mathbb{N}$ so that

$$\left\|x-\frac{\tilde{x}_{2i}-\tilde{x}_{2i-1}}{2}\right\|=\|x-Q(e_i)\|\leqslant\frac{1}{2}\quad\text{and}\quad\left\|\frac{1}{2}(\tilde{z}_{2i}-\tilde{z}_{2i-1})\right\|=1$$

which implies by Proposition 4.2 that $(x_i,f_i)_{i\in\mathbb{N}}$ has the NQP with respect to (z_i).

By construction of $(\tilde{x}_i,\tilde{f}_i)$ in Proposition 2.8 it follows that $(\tilde{f}_i)$ is semi-normalized if (f_i) has this property and since $\|Q(e_i)\pm x_i\|>\frac{1}{4}$, for $i\in\mathbb{N}$, it follows that

$$\frac{1}{4}\leqslant\|\tilde{x}_i\|\leqslant\sup_{j\in\mathbb{N}}\|x_j\|+1,$$

which implies that $(\tilde{x}_i,\tilde{f}_i)$ is semi-normalized if (x_i,f_i) has this property. □

Finally let us present an infinite dimensional argument implying that if Z is a reflexive space with basis it cannot be the associated space of a frame $(x_i,f_i)_{i\in\mathbb{N}}$, with $\|x_i\|=1$, for $i\in\mathbb{N}$, which satisfies the NQP.

Proposition 4.8. *Assume that Z is a reflexive space with a semi-normalized basis (z_i), and assume that $(x_i,(f_i))$ is a frame of an infinite dimensional Banach space X with associated space Z.*

Then $((x_i),(f_i))$ cannot have the NQP with respect to Z.

The following result follows from Proposition 4.8 as well as from Corollary 4.6.

Corollary 4.9. *A semi-normalized frame of an infinite dimensional Hilbert space H cannot have the NQP with respect to the associated Hilbert space $\ell_2(\mathbb{N})$.*

Proof. Proof of Proposition 4.8 We assume w.l.o.g. that (z_i) is bimonotone and let $T:X\to Z$ and $S:Z\to X$ be the associated decomposition and reconstruction operators, respectively.

For $C<\infty$ and $\delta>0$ define

$$\mathcal{B}_{(C,\delta)}=\left\{\sum_{i=1}^\infty\delta k_i z_i\in Z:\ (k_i)\subset\mathbb{Z},\ \left\|\sum_{i=1}^\infty\delta k_i z_i\right\|\leqslant C\right\}.$$

Assume that (x_i, f_i) has the NQP with respect to Z. Then we can choose $\delta > 0$ small enough and $C \geqslant 1$ large enough so that $S(B_{(C,\delta)})$ is ε-dense in B_X for some $0 < \varepsilon < 1$.

Since (z_i) is semi-normalized and Z is reflexive, $\mathcal{B}_{(C,\delta)}$ is weakly compact. Indeed, assume that for $n \in \mathbb{N}$,

$$y_n = \sum_{i=1}^{\infty} \delta k_i^{(n)} z_i \in \mathcal{B}_{(C,\delta)}.$$

After passing to subsequence we can assume that for all $i \in \mathbb{N}$ there is a $k_i \in \mathbb{N}$ so that $k_i^{(n)} = k_i$ whenever $n \geqslant i$. Thus, by bimonotonicity, it follows that $\|\sum_{i=1}^{n} \delta k_i z_i\| \leqslant C$, for all $n \in \mathbb{N}$, and, thus, since (z_i) is boundedly complete $\sum_{i=1}^{\infty} \delta k_i z_i \in Z$ and $\|\sum_{i=1}^{\infty} \delta k_i z_i\| \leqslant C$. Thus, $\sum_{i=1}^{\infty} \delta k_i z_i$ in $B_{(C,\delta)}$, and since $k_i^{(n)}$ converges point-wise to (k_i) and (z_i) is shrinking it is the weak limit of y_n. The support of each element in $B_{(C,\delta)}$ is finite since (z_i) is a semi-normalized basis, and thus $\mathcal{B}_{(C,\delta)}$ is countable. Since $S(\mathcal{B}_{(C,\delta)})$ is ε-dense in X, it follows that the map

$$E : X^* \to C(\mathcal{B}_{(C,\delta)}), \quad \text{with } E(x^*)\Big(\sum \delta k_i z_i\Big) = \sum \delta k_i S^*(x^*)(z_i),$$

is an isomorphic embedding (here $C(\mathcal{B}_{(C,\delta)})$ denotes the space of continuous functions on the (compact and metric) space $\mathcal{B}_{(C,\delta)}$ endowed with the weak topology). Indeed for $x^* \in B_X^*$ there is an $x \in B_X$ so that $|x^*(x)| = 1$ and a sequence $(k_i) \in \mathbb{Z} \cap c_{00}$ so that $\|x - \sum \delta k_i x_i\| \leqslant \varepsilon$, and thus

$$\big\|E(x^*)\big\| \geqslant \Big|E(x^*)\Big(\sum \delta k_i z_i\Big)\Big| = x^*\Big(\sum \delta k_i x_i\Big) = 1 + x^*\Big(\sum \delta k_i x_i - x\Big) \geqslant 1 - \varepsilon.$$

But this would mean that X^* is isomorphic to a subspace of the space of continuous functions on a countable compact space, and, thus, hereditarily c_0, which is impossible since X is a quotient of a reflexive space and thus also reflexive. □

5. Quantization and cotype

In this section we consider a quantization concept for Schauder frames, which is independent of an associated space.

Definition 5.1. Let $(x_i, f_i)_{i\in\mathbb{J}}$ be a frame of a (finite or infinite dimensional) Banach space X, $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \ldots, N\}$, for some $N \in \mathbb{N}$, and let $0 < \varepsilon$, $0 < \delta \leqslant 1$ and $1 \leqslant C < \infty$. We say that $(x_i, f_i)_{i\in\mathbb{J}}$ satisfies the (ε, δ, C)-*Bounded Coefficient Net Quantization Property* or (ε, δ, C)-BCNQP if for all $(a_i)_{i\in\mathbb{J}} \in [-1, 1]^{\mathbb{J}} \cap c_{00}(\mathbb{J})$ there is a $(k_i)_{i\in\mathbb{J}} \in \mathbb{Z}^{\mathbb{J}} \cap c_{00}(\mathbb{J})$ so that

$$\Big\|\sum_{i\in\mathbb{J}} a_i x_i - \sum_{i\in\mathbb{J}} \delta k_i x_i\Big\| \leqslant \varepsilon \quad \text{and} \quad \max_{i\in\mathbb{J}} |k_i| \leqslant \frac{C}{\delta}.$$

Remark 5.2. Let $(x_i, f_i)_{i\in\mathbb{J}}$ be a frame of X and let $0 < \varepsilon$, $0 < \delta \leqslant 1$ and $1 \leqslant C < \infty$.

(a) Since for any $(a_i)_{i\in\mathbb{J}} \in c_{00}(\mathbb{J})$ and any $i \in \mathbb{J}$ we can write $a_i = m_i\delta + \tilde{a}_i$ with $m_i \in \mathbb{N}$, $|m_i|\delta \leqslant |a_i|$ and $|\tilde{a}_i| \leqslant \delta$, for $i \in \mathbb{J}$, (x_i, f_i) satisfies the (ε, δ, C)-BCNQP implies that

for all $(a_i)_{i\in\mathbb{J}} \in c_{00}(\mathbb{J})$ there is a $(k_i)_{i\in\mathbb{J}} \in \mathbb{Z}^{\mathbb{J}} \cap c_{00}(\mathbb{J})$ so that

$$\Big\|\sum_{i\in\mathbb{J}} a_i x_i - \sum_{i\in\mathbb{J}} \delta k_i x_i\Big\| \leqslant \varepsilon \quad \text{and} \quad \max_{i\in\mathbb{J}} |k_i| \leqslant \max_{i\in\mathbb{J}} |a_i| + \frac{C}{\delta}. \tag{26}$$

(a) immediately implies.

(b) If (x_i, f_i) satisfies (ε, δ, C)-BCNQP and $0 < \lambda \leqslant 1$ then (x_i, f_i) satisfies $(\lambda\varepsilon, \lambda\delta, 1 + \lambda C)$-BCNQP.

(c) If (x_i) is a semi-normalized basis of X and (f_i) are the coordinate functionals with respect to (x_i) and (x_i) satisfies the (ε, δ)-NQP (Definition 3.1), then (x_i, f_i) satisfies the (ε, δ, C)-BCNQP with $C = 1 + \varepsilon \sup_{i\in\mathbb{J}} \|f_i\|$. Indeed, for $x \in X$, $x = \sum_{i=1} a_i x_i$, with $|a_i| \leqslant 1$, there is a sequence $(k_i) \in \mathbb{Z}$ with finite support so that $\|x - \sum \delta k_i x_i\| \leqslant \varepsilon$ and

$$\delta \max |k_i| \leqslant \max_i \Big(\big|f_i(x)\big| + \Big|f_i\Big(x - \sum \delta k_i x_i\Big)\Big|\Big) \leqslant 1 + \varepsilon \sup_{i\in\mathbb{J}} \|f_i\|.$$

We will connect the property BCNQP with properties of the *cotype* of the Banach space.

Definition 5.3. Let $p \leqslant 2$. We say that a Banach space X *has type* p if there is a $c < \infty$ so that for all $n \in \mathbb{N}$ and all vectors $x_1, x_2, \ldots, x_n \in X$,

$$\bigg(\text{ave}\Big\|\sum_{i=1}^{n} \pm x_i\Big\|^2\bigg)^{1/2} = \bigg(2^{-n} \sum_{(\sigma_i)_{i=1}^n \in \{\pm 1\}^n} \Big\|\sum_{i=1}^{n} \sigma_i x_i\Big\|^2\bigg)^{1/2} \leqslant c\bigg(\sum_{i=1}^{n} \|x_i\|^p\bigg)^{1/p}.$$

In that case the smallest such c will be denoted by $T_p(X)$.

Let $q \geqslant 2$. We say that a Banach space X *has cotype* q if there is a $c < \infty$ so that for all $n \in \mathbb{N}$ and all $x_1, x_2, \dots, x_n \in X$:

$$\left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q} \leqslant c\left(\text{ave}\left\|\sum_{i=1}^n \pm x_i\right\|^2\right)^{1/2} = c\left(2^{-n}\sum_{(\sigma_i)_{i=1}^n \in \{\pm 1\}^n}\left\|\sum_{i=1}^n \sigma_i x_i\right\|^2\right)^{1/2}.$$

The smallest of all these constants will be denoted by $C_q(X)$.

We say that X has only trivial type, or only trivial cotype if $T_p(X) = \infty$ for all $p > 1$, or $C_q(X) = \infty$, for all $q < \infty$.

Basic properties of spaces with type and cotype can be found for example in [7] or [28]. We are mainly interested in estimates of the volume ratio of the unit ball B_X of a finite dimensional space X using $C_q(X)$ and the connection between finite cotype and the property of containing ℓ_∞^n's uniformly.

Assume X is an n-dimensional space which we identify with $(\mathbb{R}^n, \|\cdot\|)$. Let E be the John ellipsoid of the unit ball B_X of X, i.e. the ellipsoid contained in B_X having maximal volume. It was show in [18] (see also [28, Chapter 3]) that E is unique. We call the ratio $\mathrm{Vol}^{1/n}(B_X)/\mathrm{Vol}^{1/n}(E)$ the *volume ratio of* B_X. Combining [29, Theorem 6], which establishes an upper estimate for the volume ratio using $T_p(X^*)$, with a result of Maurey and Pisier [22,23] (see also [7, Proposition 13.17]) estimating $T_p(X^*)$ and a result of Pisier ([27] (see also [28, Theorem 2.5]) estimating the K-convexity constant $K(X)$ of X, we obtain the connection between the volume ratio of B_X and the cotype constant of X.

Theorem 5.4. *There is a universal constant d so that for all finite dimensional Banach spaces X, with $n = \dim(X) \geqslant 2$, and all $2 \leqslant q < \infty$,*

$$\left(\frac{\mathrm{Vol}(B_X)}{\mathrm{Vol}(E)}\right)^{1/n} \leqslant d C_q(X) n^{\alpha(q)} \ln n, \tag{27}$$

where $E \subset X$ is the John ellipsoid of B_X and $\alpha(q) := \frac{1}{2} - \frac{1}{q}$.

We will also need a second upper estimate for the volume ratio due to Milman and Pisier [24].

Theorem 5.5. *(See [24], and see also [28, Theorem 10.4].) There is a universal constant A so that for any finite dimensional Banach space X,*

$$\left(\frac{\mathrm{Vol}(B_X)}{\mathrm{Vol}(E)}\right)^{1/n} \leqslant g\big(C_2(X)\big) := A C_2(X) \ln\big(1 + C_2(X)\big) \tag{28}$$

where $E \subset X$ is the John ellipsoid of B_X.

The next result describes the connection between the property of having a finite cotype for $q < \infty$ and the property of containing ℓ_∞^n's uniformly.

Theorem 5.6. *(See [22,23].) For $N \in \mathbb{N}$ there is a $q(N) \in (2, \infty)$ and a $C(N) < \infty$ so that*:

For any (finite or infinite dimensional) Banach space X which does not contain a 2-isomorphic copy of ℓ_∞^N we have that $C_{q(N)}(X) \leqslant C(N)$. (29)

Finally we will need the following result from [25]. It is implicitly already contained in [12, pp. 95–97], and it has probably been known for much longer.

In order to state it we will need the following notation. Let $m \leqslant n \in \mathbb{N}$ and let $L \subset \mathbb{R}^n$ be an m-dimensional subspace. Let Q_n be the unit cube in $\mathbb{R}^n$. By a simple compactness argument there is a projection $P : \mathbb{R}^n \to L$ for which $\mathrm{Vol}(P(Q_n))$ is minimal. In that case we call the image $P(Q_n)$ a *minimal-volume projection of Q_n onto L.*

Theorem 5.7. *(See [25, Theorem 1].) Let L be a linear subspace of $\mathbb{R}^n$, and let $\mathcal{M}$ be the set of all minimal volume projections of Q_n onto L.*

Then $\mathcal{M}$ contains a parallelepiped.

We are now in the position to state and to prove the connection between cotype and BCNQP in the finite dimensional case.

Theorem 5.8. *There is a map $n_0 : [1,\infty)^2 \to [0,\infty)$ so that for all finite dimensional Banach spaces X the following holds.*

If $(x_i, f_i)_{i=1}^N$ is a frame of X, with $\|x_i\| = 1$, for $i \leqslant N$, and which satisfies $(1,\delta,C)$-BCNQP for some $0 < \delta < 1$ and $C \geqslant 1$, then for all $2 \leqslant q < \infty$

$$N \geqslant \dim(X)\ln\bigl(\dim(X)\bigr)\frac{1}{2q\ln(1+2\frac{C}{\delta})}, \quad \textit{whenever } \dim(X) \geqslant n_0\biggl(\frac{C}{\delta}, KC_q(X)\biggr),$$

where K is the projection constant of $(x_i, f_i)_{i=1}^N$.

Proof. Let Z be the space with a basis (z_i) and let $T : X \to Z$ and $S : Z \to X$ be the associated decomposition and the reconstruction operator as constructed in the proof of Proposition 2.4(a) "$\Rightarrow$." Since the x_i's are normalized the z_i's are also of norm 1. After a linear transformation we can assume that $Z = \mathbb{R}^N$ and $(z_i)_{i=1}^N$ is the unit vector basis of $\mathbb{R}^N$. Since (z_i) is a bimonotone basis it follows that $\|z_i^*\| = 1$, for $i \leqslant N$. Hence $B_Z \subset Q_N$, where Q_N denotes the unit cube in $\mathbb{R}^N$. Define $L = T(X)$ and put $n = \dim(X) = \dim(L)$. Since $S \circ T = \mathrm{Id}_X$ it follows that $P = T \circ S$ is a projection from Z onto L and if we denote the John ellipsoid of $T(B_X)$ by E and we deduce that (recall that by Proposition 2.4(a) $\|T\| \leqslant K$)

$$E \subset T(B_X) = P \circ T(B_X) \subset P(K \cdot B_Z) \subset P(K \cdot Q_N). \tag{30}$$

By Theorem 5.7 there is a minimal-volume projection M of Q_N onto L which is a parallelepiped. Let B_n denote the n-dimensional Euclidean ball in $\mathbb{R}^n$. Since there is a universal constant c so that

$$\mathrm{Vol}(B_n) \leqslant \biggl(\frac{c}{\sqrt{n}}\biggr)^n,$$

and since

$$\frac{1}{K}E \subset \frac{1}{\|T\|}E \subset L \cap B_Z \subset L \cap Q_N \subset M,$$

we deduce from the fact that B_n is the John ellipsoid of the unit cube in $\mathbb{R}^n$ [18] (see also [28, Chapter 3]), that

$$K\biggl(\frac{\mathrm{Vol}(P(Q_N))}{\mathrm{Vol}(E)}\biggr)^{1/n} = \biggl(\frac{\mathrm{Vol}(P(Q_N))}{\mathrm{Vol}(\frac{1}{K}E)}\biggr)^{1/n} \geqslant \biggl(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(\frac{1}{K}E)}\biggr)^{1/n} \geqslant \frac{\sqrt{n}}{c}.$$

The last inequality follows from applying a linear transformation A to L so that $A(M)$ is a the unit cube in L (with respect to some orthonormal basis of L) and, thus, $A(\frac{1}{K}E)$ is an ellipsoid whose volume cannot exceed that of the Euclidean unit ball in L. Since $T : (X, \|\cdot\|) \to (L, \|\cdot\|_{T(B_X)})$, where $\|\cdot\|_{T(B_X)}$ is the Minkowski functional for $T(B_X)$, is an isometry it follows from Theorem 5.4 that

$$\begin{aligned}\mathrm{Vol}^{1/n}\bigl(T(B_X)\bigr) &\leqslant dC_q(X)n^{\alpha(q)}\ln(n)\,\mathrm{Vol}^{1/n}(E)\\ &\leqslant dcKC_q(X)n^{-\frac{1}{q}}\ln(n)\,\mathrm{Vol}^{1/n}\bigl(P(Q_N)\bigr)\end{aligned} \tag{31}$$

(the universal constant d was introduced in Theorem 5.4).

Since the zonotope

$$P(Q_N) = T \circ S\Biggl(\Biggl\{\sum_{i=1}^N a_i z_i\colon\ |a_i| \leqslant 1\Biggr\}\Biggr) = \Biggl\{\sum_{i=1}^N a_i T(x_i)\colon\ |a_i| \leqslant 1\Biggr\}$$

contains at most $(1+\frac{2C}{\delta})^N$ points from the set $D = \{\sum \delta n_i T(x_i)\colon\ (n_i) \subset \mathbb{Z},\ \max \delta|n_i| \leqslant C\}$ and since from our assumption that $(x_i f_i)_{i=1}^N$ satisfies the $(1,\delta,C)$-BCNQP it follows that

$$P(Q_N) \subset \bigcup_{z\in D} z + T(B_X),$$

we deduce that

$$\biggl(1+\frac{2C}{\delta}\biggr)^N \geqslant \frac{\mathrm{Vol}(P(Q_N))}{\mathrm{Vol}(T(B_X))} \geqslant \biggl(\frac{n^{1/q}}{KdcC_q(X)\ln(n)}\biggr)^n$$

and, thus,

$$N \geqslant \frac{n\ln(n)}{q\ln(1+\frac{C}{\delta})} - \frac{n\ln(\ln(n))}{\ln(1+\frac{2C}{\delta})} - \frac{n\ln(dcKC_q(X))}{\ln(1+\frac{2C}{\delta})},$$

which easily implies our claim. $\square$

In the next result we will show that, up to a constant factor, the result in Theorem 5.8 is sharp. We are using the simple fact that for any number $0 \leqslant r \leqslant 1$ and any $m \in \mathbb{N}$, r can be approximated by a finite sum of dyadic numbers, say $\tilde{r} = \sum_{j=1}^m \sigma_j 2^{-j}$, $\sigma_j \in \{0,1\}$, for $j = 1,\dots,m$, so that $|r - \tilde{r}| \leqslant 2^{-m}$.

Proposition 5.9. *Let X be an n-dimensional space with an Auerbach basis $(e_i, e_i^*)_{i=1}^n$. Let $m \in \mathbb{N}$ and let K be the projection constant of $(e_i)_{i=1}^n$. Then there is a frame $(x_{(i,j,s)}, f_{(i,j,s)}\colon 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ s = 0,1)$ (ordered lexicographically) so that*

$$\frac{1}{2} \leqslant \|x_{(i,j,s)}\| \leqslant 2 \quad \text{and} \quad \|f_{(i,j,s)}\| = 1, \quad \text{if } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m \text{ and } s = 0,1, \tag{32}$$

$$\forall (a_{(i,j,s)}\colon 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ s = 0,1) \subset [-1,1]$$
$$\exists (k_{(i,j,s)}\colon i \leqslant n,\ j \leqslant m,\ s = 0,1) \subset \{-3,-2,\dots,3\}$$
$$\left\| \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0}^1 a_{(i,j,s)} x_{(i,j,s)} - \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0}^1 k_{(i,j,s)} x_{(i,j,s)} \right\| \leqslant 1 + n \frac{2^{-m}}{1-2^{-m}} \tag{33}$$

$\left(\text{i.e. } (x_{(i,j,s)}, f_{(i,j,s)}\colon 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ s = 0,1) \text{ satisfies the } \left(1 + n\frac{2^{-m}}{1-2^{-m}}, 1, 3\right)\text{-BCNQP}\right).$

The projection constant of $(x_{(i,j,s)}, f_{(i,j,s)}\colon 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ s = 0,1)$ does not exceed $4K$. (34)

Proof. For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ define $x_{(i,j,0)} = e_1$, $x_{(i,j,1)} = e_1 + \frac{2^{-j}}{1-2^{-m}} e_i$, $f_{(i,j,0)} = -e_i^*$ and $f_{(i,j,1)} = e_i^*$. Since for every $x \in X$

$$x = \sum_{i=1}^n e_i^*(x) e_i = \sum_{i=1}^n \sum_{j=1}^m e_i^*(x) e_i \frac{2^{-j}}{1-2^{-m}} = \sum_{i=1}^n \sum_{j=1}^m f_{(i,j,1)}(x) x_{(i,j,1)} + f_{(i,j,0)}(x) x_{(i,j,0)},$$

$(x_{(i,j,s)}, f_{(i,j,s)}\colon 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ s = 0,1)$ is a frame of X and it satisfies (32). In order to verify (33) let $(a_{(i,j,s)}\colon 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ s = 0,1) \subset [-1,1]$ be given. For $i = 1,2,\dots,n$ it follows that $|\sum_{j=1}^m a_{(i,j,1)} \frac{2^{-j}}{1-2^{-m}}| \leqslant 1$, and, thus, we can choose $(k_{(i,j,1)}\colon j \leqslant m) \subset \{0, \pm 1\}$ so that for each $i \leqslant n$

$$\left| \sum_{j=1}^m a_{(i,j,1)} \frac{2^{-j}}{1-2^{-m}} - \sum_{j=1}^m k_{(i,j,1)} \frac{2^{-j}}{1-2^{-m}} \right| \leqslant \frac{2^{-m}}{1-2^{-m}}. \tag{35}$$

Since the absolute value of $M = \sum_{i=1}^n \sum_{j=1}^m a(i,j,1) + a(i,j,0) - k_{(i,j,1)}$ is at most $3nm$ we can choose for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, $k_{(i,j,0)} \in \{-3,-2,\dots,2,3\}$ so that $a = M - \sum_{i=1}^n \sum_{j=1}^m k_{(i,j,0)}$, has absolute value at most 1. We compute

$$\begin{aligned}
&\left\| \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0}^1 a_{(i,j,s)} x_{(i,j,s)} - \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0}^1 k_{(i,j,s)} x_{(i,j,s)} \right\| \\
&\quad \leqslant \left\| \sum_{i=1}^n \sum_{j=1}^m a_{(i,j,1)} e_i \frac{2^{-j}}{1-2^{-m}} - \sum_{i=1}^n \sum_{j=1}^m k_{(i,j,1)} e_i \frac{2^{-j}}{1-2^{-m}} \right\| \\
&\qquad + \left| \sum_{i=1}^n \sum_{j=1}^m a_{(i,j,1)} + a_{(i,j,0)} - k_{(i,j,0)} - k_{(i,j,1)} \right| \\
&\quad \leqslant 1 + \sum_{i=1}^n \left| \sum_{j=1}^m (a_{(i,j,1)} - k_{(i,j,1)}) \frac{2^{-j}}{1-2^{-m}} \right| \leqslant 1 + n \frac{2^{-m}}{1-2^{-m}}
\end{aligned}$$

which proves (33).

To estimate the projection constant of $(x_{(i,j,s)}, f_{(i,j,s)}\colon 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ s = 0,1)$ we denote by $\leqslant_{\text{lex}}$ the lexicographic order on $\{(i,j,s)\colon i \leqslant n,\ j \leqslant m,\ s = 0,1\}$, and let

$$x = \sum_{i=1}^n a_i e_i = \sum_{i=1}^n \sum_{j=1}^m -a_i e_1 + a_i e_1 + a_i e_i \frac{2^{-j}}{1-2^{-m}} = \sum_{i=1}^n \sum_{j=1}^m \sum_{s=0,1} f_{(i,j,s)}(x) x_{(i,j,s)}$$

and $(i_0, j_0, s_0) \leqslant_{\text{lex}} (i_1, j_1, s_1)$. Then, if $i_0 < i_1$,

$$\begin{aligned}
&\left\| \sum_{(i_0,j_0,s_0) \leqslant_{\text{lex}} (i,j,s) \leqslant_{\text{lex}} (i_1,j_1,s_1)} f_{(i,j,s)}(x) x_{(i,j,s)} \right\| \\
&\quad = \left\| 1_{\{s_0=1\}} \left[a_{i_0} e_1 + a_{i_0} \frac{2^{-j_0}}{1-2^{-m}} \right] + \sum_{j=j_0+1}^m -a_{i_0} e_1 + a_{i_0} e_1 + a_{i_0} \frac{2^{-j}}{1-2^{-m}} e_{i_0} \right.
\end{aligned}$$

$$+\sum_{i=i_0+1}^{i_1-1}\sum_{j=1}^{m}\left(-a_ie_1+a_ie_1+a_i\frac{2^{-j}}{1-2^{-m}}e_i\right)+\sum_{j=1}^{j_1}\left(-a_{i_1}e_1+a_{i_1}e_1+a_{i_1}\frac{2^{-j}}{1-2^{-m}}e_{i_1}\right)-1_{\{s_1=0\}}a_{i_1}e_1\Bigg\|$$

$$\leqslant 2|a_{i_0}|+\left\|\sum_{i=i_0+1}^{i_1-1}a_ie_i\right\|+|a_{i_1}|\leqslant 4K\|x\|.$$

If $i_0=i_1$ similar estimates give the to the same result for the remaining cases and (34) follows. □

Remark 5.10. If we choose in Proposition 5.9 $m=\lceil 2\log n\rceil$ and thus $2^{-m}\simeq 1/n^2$ we obtain a frame for X of approximate size $4n\log_2(n)$ having the $(3,1,3)$-BCNQP. Thus as we mentioned earlier, up to a constant Theorem 5.8 is best possible.

Remark 5.11. In Theorem 5.8 we assumed for simplicity that the x_i's of our frame are normalized. It is easy to see that the same proof works for a general frame, in that case n_0 depends also on $a=\min\{\|x_i\|:\ i\leqslant N,\ x_i\neq 0\}$ and $b=\max\{\|x_i\|:\ i\leqslant N\}$.

With a similar proof to that of Theorem 5.8 we derive an upper estimate for $\min_{i\leqslant N}\|x_i\|$, $i\leqslant N$, assuming that $(x_i,f_i)_{i=1}^N$ is a frame of an n-dimensional space X which satisfies the $(1,\delta,C)$-BCNQP for some choice of $\delta>0$ and $C<\infty$ assuming that N is proportional to n.

Theorem 5.12. *For any choice of $\delta\in(0,1]$, and $C,K,q,c_2\geqslant 1$ there is a value $h=h(\delta,C,K,q,c_2)$ so that the following holds for all $n\in\mathbb{N}$.*

If X is an n-dimensional space, $N\leqslant qn$ and $(x_i,f_i)_{i=1}^N$ is a frame of X with projection constant K which has the $(1,\delta,C)$-BCNQP, then if $C_2(X)\leqslant c_2$,

$$\min_{i\leqslant N}\|x_i\|\leqslant\frac{h(\delta,C,K,q,c_2)}{\sqrt{n}}.$$

Sketch of proof. Let $(x_i,f_i)_{i=1}^N$ be a frame of X, $N\leqslant qn$, which has the $(1,\delta,C)$-BCNQP and projection constant K. As in the proof of Theorem 5.8 we let Z be the associated space with basis (z_i) which was constructed in Proposition 2.4, $T:X\to Z$ the associated decomposition operator, and S the associated reconstruction operator. Let $L=T(X)$, and $P=T\circ S$, and let us also assume that $Z=\mathbb{R}^N$ and $z_i=e_i$ for $i\leqslant N$. Note that now $\|z_i\|=\|x_i\|$ and $z_i^*=\|x_i\|^{-1}$ and we can therefore follow the proof of Theorem 5.8 replacing Q_N by the box

$$\tilde{Q}_N=\prod_{i=1}^N\left[-\frac{1}{\|x_i\|},\frac{1}{\|x_i\|}\right].$$

As in the proof of Theorem 5.8 it follows that $\frac{1}{K}T(B_X)\subset P(B_Z)\subset P(\tilde{Q}_N)$. For the John ellipsoid E of $T(B_X)$ it follows therefore that $\frac{1}{K}E\subset M$, where a M is a minimal volume projection of $\tilde{Q}_N$ which is also a parallelepiped in L, and as before we deduce that $K\,\mathrm{Vol}^{1/n}(P(\tilde{Q}_N))\geqslant\mathrm{Vol}^{1/n}(E)\sqrt{n}/c$. Instead of applying Theorem 5.4 we now use Theorem 5.5 and letting $\alpha=\min_{i\leqslant N}\|x_i\|$ we deduce that

$$\begin{aligned}\mathrm{Vol}^{1/n}\big(T(B_X)\big)&\leqslant g\big(C_2(X)\big)\,\mathrm{Vol}^{1/n}(E)\\&\leqslant\frac{g(C_2(X))cK\,\mathrm{Vol}^{1/n}(P(\tilde{Q}_N))}{\sqrt{n}}\leqslant\frac{g(C_2(X))cK\,\mathrm{Vol}^{1/n}(P(Q_N))}{\sqrt{n}\alpha}.\end{aligned}$$

We can again compare the volume of the zonotope $P(Q_N)$ with the volume of the union $\bigcup_{z\in D}z+T(B_X)$, where D is defined as in the proof of Theorem 5.4, and deduce that

$$\left(1+\frac{2C}{\delta}\right)^{qn}\geqslant\left(1+\frac{2C}{\delta}\right)^{N}\geqslant\frac{\mathrm{Vol}(P(Q_N))}{\mathrm{Vol}(T(B_X))}\geqslant\left(\frac{\sqrt{n}\alpha}{g(C_2(X))cK}\right)^n.$$

Taking the nth root on both sides yields our claim. □

Remark 5.13. In Section 6 we will recall a result of Lyubarskii and Vershinin [21] which shows that for $q>1$ there are $\varepsilon<1$, $\delta<1$, $C<\infty$ so that for any $n\in\mathbb{N}$ and there is a Hilbert frame $(x_i)_{i=1}^N$ of ℓ_2^n, with $N\leqslant qn$, so that $(x_i)_{i=1}^N$ satisfies (ε,δ,C)-BCNQP.

As in the previous section we formulate a corollary of Theorem 5.8 for the infinite dimensional situation.

Corollary 5.14. *Assume that X has the π_λ-property and that $\overline{P}=(P_n)$ is a sequence of uniformly bounded projections approximating point-wise the identity on X. Let (x_i, f_i) be a frame of X, with (x_i) being bounded, and assume for an increasing sequence $(L_n)\subset\mathbb{N}$, $0<r,\delta,\varepsilon<1$ and $C<\infty$ that the following conditions are satisfied:*

(a) *for $n\in\mathbb{N}$ and $x\in X_n=P_n(X)$*

$$\left\| x-\sum_{i=1}^{L_n}\langle f_i,x\rangle P_n(x)\right\| \leqslant (1-r)\|x\|,$$

(b) *$\|P_n(x_i)\|\geqslant r$, for all $n\in\mathbb{N}$ and $i\leqslant L_n$, and*

(c) *for all $n\in\mathbb{N}$ and all $x\in\{\sum_{i=1}^{L_n}a_ix_i\colon |a_i|\leqslant 1$ for $i=1,2,\ldots,L_n\}$ there is a sequence $(k_i)_{i\leqslant L_n}$ so that*

$$\left\| x-\sum_{i=1}^{L_n}\delta k_i x_i\right\| <\varepsilon \quad \text{and} \quad \max_{i\leqslant L_n}|k_i|\delta\leqslant C$$

(*in particular (x_i, f_i) satisfies the BCNQP*).

Then there is either a constant $c>0$ so that $L_n\geqslant c\dim P_n(X)\ln(\dim P_n(X))$ or the spaces $\{\ell_\infty^n\colon n\in|N\}$ are uniformly contained in X.

Proof. From assumption (a) it follows that the operators (Q_n), with

$$Q_n: X_n\to X_n,\qquad x\mapsto \sum_{i=1}^{L_n}\big\langle P_n^*(f_i),x\big\rangle P_n(x_i),$$

are uniformly bounded ($\|Q_n\|\leqslant 2$, for $n\in\mathbb{N}$), invertible and their inverses are uniformly bounded ($\|Q_n^{-1}\|\leqslant\frac{1}{r}$, for $n\in\mathbb{N}$). For $x\in X_n$ we write

$$x=Q_n^{-1}Q_n(x)=\sum_{i=1}^{L_n}\langle f_i,x\rangle\big(Q_n^{-1}\circ P_n\big)(x_i),$$

and deduce therefore that

$$\big(y_i^{(n)},g_i^{(n)}\big)_{i=1}^{L_n}:=\big(\big(Q_n^{-1}\circ P_n\big)(x_i),\,f_i|_{X_n}\big)_{i=1}^{L_n}$$

is a frame of X_n. We now verify that for $n\in\mathbb{N}$ $(y_i^{(n)},g_i^{(n)})_{i=1}^{L_n}$ satisfies the $(\tilde\varepsilon,\delta,C)$-BCNQP for some $\tilde\varepsilon>0$, which is independent of n. Indeed by assumption (c) one can choose for $n\in\mathbb{N}$ and $(a_i)_{i=1}^{L_n}\in[-1,1]$ some $(k_i)_{i=1}^{L_n}\subset\mathbb{Z}$ so that

$$\left\|\sum_{i=1}^{L_n}a_ix_i-\sum_{i=1}^{L_n}\delta k_ix_i\right\|\leqslant\varepsilon \quad\text{and}\quad \max_{i\leqslant L_n}|k_i|\leqslant\frac{C}{\delta}$$

and, thus,

$$\left\|\sum_{i=1}^{L_n}a_iy_i^{(n)}-\sum_{i=1}^{L_n}\delta k_iy_i^{(n)}\right\|=\left\|\sum_{i=1}^{L_n}a_i\big(Q_n^{-1}\circ P_n\big)(x_i)-\sum_{i=1}^{L_n}\delta k_i\big(Q_n^{-1}\circ P_n\big)(x_i)\right\|$$
$$\leqslant\varepsilon\max_{i\leqslant L_n}\big\|Q_n^{-1}\circ P_n\big\|\leqslant\frac{\varepsilon}{r}\sup_n\|P_n\|=:\tilde\varepsilon.$$

Then for $n\in\mathbb{N}$ and $i\leqslant L_n$ it follows from assumption (b) that

$$\frac{r}{2}\leqslant\frac{r}{\|Q_n\|}\leqslant\big\|y_i^{(n)}\big\|\leqslant\|P_n\|\cdot\big\|Q_n^{-1}\big\|\cdot\|x_i\|\leqslant\frac{\sup_j\|P_j\|\sup_j\|x_j\|}{r}<\infty.$$

Thus Theorem 5.8, Remark 5.11 and Theorem 5.6 yield our claim. □

By Remark 5.2, for semi-normalized bases (x_i) (together with their coordinate functionals) the properties BCNQP and NQP are equivalent. We therefore deduce from Theorem 5.8 the following

Corollary 5.15. *An infinite dimensional Banach space X with nontrivial cotype cannot have a semi-normalized basis having the NQP. In particular* (*see Problem* 5.18 *in* [8]) *ℓ_1 does not have a semi-normalized basis with the NQP.*

Proof. Suppose (x_i) is a semi-normalized basis with the (ε,δ)-NQP. Then we let (P_n) be the basis projections and $L_n=n$. By Corollary 5.14 X does not have finite cotype. □

6. Concluding remarks and open problems

Kashin's [20] celebrated result states that for any $\lambda > 1$ there is a $K = K_\lambda$ so that for any $n \in \mathbb{N}$ and any $N \geqslant \lambda n$, $N \in \mathbb{N}$, there is an orthogonal projection U from $\mathbb{R}^N$ onto $\mathbb{R}^n$ (i.e. U is an n by N matrix whose rows are orthonormal) so that

$$B_n \subset \frac{K}{\sqrt{N}} U(Q_N) \subset K B_n \tag{36}$$

(as before B_n is the euclidean unit ball in $\mathbb{R}^n$ while Q_N is the unit cube in $\mathbb{R}^N$).

Lyubarskii and Vershinin observed in [21] that the column vectors $(u_i)_{i=1}^N$ form a tight frame (with $A = B = 1$), that the first inclusion in (36) yields that every $x \in B_n$ can be written as

$$x = \sum_{i=1}^N \frac{K}{\sqrt{N}} a_i u_i \quad \text{with } \|(a_i)\|_{\ell_\infty} \leqslant 1,$$

and that the second inclusion implies that the operator $\frac{1}{\sqrt{N}} U : \ell_\infty^N \to \ell_2^n$ is of norm not greater than 1, and that therefore for given $\varepsilon > 0$ there is a sequence $(k_i)_{i=1}^N \subset \mathbb{Z} \cup [-K/\varepsilon, K/\varepsilon]$, so that $\max_{i \leqslant N} |a_i K - k_i \varepsilon| \leqslant \varepsilon$ and, thus,

$$\left\| x - \sum_{i=1}^N \varepsilon k_i \frac{u_i}{\sqrt{N}} \right\| \leqslant \varepsilon \quad \text{and} \quad \max |\varepsilon k_i| \leqslant K. \tag{37}$$

Thus, Kashin's orthogonal projections (which are actually chosen randomly), lead to a frame $(x_i^{(n)})_{i \leqslant N} = (u_i^{(n)}/\sqrt{N})_{i \leqslant N}$ for ℓ_2^n, whose length is not larger than a fixed multiple of n, and, for which we can represent any element x in B_n as a quantized linear combination with bounded coefficients. Since the zonotope $\{\sum_{i=1}^N a_i x_i | a_i| \leqslant 1 \text{ for } i = 1, 2, \dots, N\}$ lies in B_n, it follows that the Hilbert frame $(x_i^{(n)})_{i \leqslant N}$ satisfies for any $\varepsilon > 0$ the $(\varepsilon, \varepsilon, K)$-BCNQP.

In view of the results presented in Sections 4 and 5 this is the best one could do in the finite dimensional case. We are therefore interested in extensions of this result by Lyubarskii and Vershinin to other spaces as well as the infinite dimensional space,

Problem 6.1. Does the above cited result hold for other finite dimensional spaces? More precisely, assume that $0 < \delta, \varepsilon < 1$, $C \geqslant 1$ are fixed. For which $n \in \mathbb{N}$ and which n-dimensional spaces X can we find a frame $(x_i, f_i)_{i=1}^N$, with, say $N = 2n$, so that for any $x \in B_X$ there is a $(k_i)_{i=1}^N \subset \mathbb{Z}$ so that

$$\left\| x - \sum_{i=1}^N \delta k_i x_i \right\| \leqslant \varepsilon \quad \text{and} \quad \max |\delta k_i| \leqslant C.$$

Remark 6.2. The above presented argument from [21] shows that if there is a quotient $Q : \ell_\infty^N \to X$, and a frame $(x_i, f_i)_{i=1}^N$ of X, so that $Q(e_i) = x_i$, for $i = 1, \dots, N$, and so that for some $K < \infty$ $B_X \subset Q(B_{\ell_\infty}) \subset K B_X$, then there is for all $x \in B_X$ and all $\delta > 0$ a sequence $(k_i)_{i \leqslant N} \subset \mathbb{Z}$, so that

$$\left\| x - \sum_{i=1}^N \delta k_i x_i \right\| \leqslant \|Q\| \delta/2 \leqslant K\delta/2 \quad \text{and} \quad \max_{i \leqslant N} |k_i| \leqslant \frac{1}{\delta}.$$

Conversely, assume that for some $0 < \delta, \varepsilon < 1$, $C \geqslant 1$ we can find for all $x \in B_X$ a sequence $(k_i) \subset \mathbb{Z}$ so that

$$\left\| x - \sum_{i=1}^N \delta k_i x_i \right\| \leqslant \varepsilon \quad \text{and} \quad \max_{i \leqslant N} |k_i| \leqslant \frac{C}{\delta}.$$

Then we can choose by induction for $x \in B_X$ a z_i with

$$z_n = \sum_{i=1}^N k_i^{(n)} \delta x_i, \, (k_i) \subset \mathbb{Z} \cap [-C/\delta, C/\delta],$$

so that

$$\left\| x - \sum_{i=1}^n \varepsilon^{i-1} z_i \right\| \leqslant \varepsilon^n.$$

Indeed assuming $z_1, \dots, z_{n-1}$ have been chosen we apply our assumption to $y = \varepsilon^{1-n}[x - \sum_{i=1}^{n-1} \varepsilon^{1-i} z_i] \in B_X$ to find z_n.

Thus, it follows that

$$x = \sum_{i=1}^{\infty} \varepsilon^{i-1} z_i = \sum_{j=1}^{N} x_j \sum_{i=1}^{\infty} \varepsilon^{i_1} \delta k_j^{(i)},$$

which means that there is a $C_1 < \infty$ only depending on ε, δ and C so that

$$B_X \subset \left\{ \sum_{j=1}^{N} a_i x_i \colon |a_i| \leqslant C_1 \right\}.$$

If we define now $Q : \ell_\infty^N \to B_X$, $z \mapsto \sum_{i=1}^N C_1 z_i x_i$, and deduce that Q is a quotient map and that $B_X \subset Q(B_{\ell_\infty})$ but we cannot deduce (at least not obviously) a bound for $\|Q\|$.

Problem 6.3. Is there an infinite dimensional version of the result of Lyubarskii and Vershinin? I.e. for which infinite dimensional Banach spaces X with a basis (e_i) does there exist $0 < \delta, \varepsilon < 1$, $C \geqslant 1$ and a frame $(x_i, f_i)_{i \in \mathbb{N}}$ so that for any $x \in B_X$, $n = \max \operatorname{supp}(x) < \infty$, there is a $(k_i)_{i=1}^N$, with, say, $N \leqslant 2n$, so that

$$\left\| x - \sum_{i=1}^{N} \delta k_i x_i \right\| \leqslant \varepsilon \quad \text{and} \quad \max |\delta k_i| \leqslant C?$$

References

[1] A. Alroubi, K. Gröchenig, Nonuniform sampling and reconstruction in shift invariant spaces, SIAM Rev. 43 (2001) 585–620.
[2] P.G. Casazza, The art of frame theory, Taiwanese J. Math. 4 (2) (2000) 129–201.
[3] P.G. Casazza, Approximation properties, in: William B. Johnson, Joram Lindenstrauss (Eds.), Handbook on the Geometry of Banach Spaces, vol. 1, North-Holland, Amsterdam, 2001, pp. 271–316.
[4] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, 2003.
[5] P.G. Casazza, D. Han, D.R. Larson, Frames for Banach spaces, in: The Functional and Harmonic Analysis of Wavelets and Frames, San Antonio, TX, 1999, in: Contemp. Math., vol. 247, 1999, pp. 149–182.
[6] I. Daubechies, R. DeVore, Approximating a bandlimited function using very coarsely quantized data: A family of stable sigma-delta modulators of arbitrary order, Ann. of Math. 158 (2003) 679–710.
[7] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, Cambridge Stud. Adv. Math., vol. 43, Cambridge University Press, Cambridge, 1995, xvi+474 pp.
[8] S.J. Dilworth, E. Odell, Th. Schlumprecht, András Zsák, Coefficient quantization in Banach spaces, Found. Comp. Math., in press.
[9] D.L. Donoho, M. Elad, Optimally sparse representations in general (non orthogonal dictionaries via ℓ_1 minimization, Proc. Natl. Acad. Sci. USA 100 (2003) 2197–2202.
[10] Y.C. Eldar, G.D. Forney, Optimal tight frames and quantum measurement, IEEE Trans. Inform. Theory 48 (2002) 599–610.
[11] M. Fabian, P. Habala, P. Hajek, V. Montesinos Santalucía, J. Pelant, V. Zizler, Functional Analysis and Infinite-Dimensional Geometry, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC8, Springer-Verlag, New York, 2001, x+451 pp.
[12] Y. Gordon, M. Meyer, A. Pajor, Ratios of volumes and factorization through l_∞, Illinois J. Math. 40 (1) (1996) 91–107.
[13] V.K. Goyal, J. Kovačević, J.A. Kelner, Quantized frame expansions with erasures, Appl. Comput. Harmon. Anal. 10 (2001) 203–233.
[14] K. Gröchenig, Describing functions: Atomic decompositions versus frames, Monatsh. Math. 112 (1991) 1–41.
[15] K.H. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2000.
[16] K.H. Gröchenig, C. Heil, Modulation spaces and pseudodifferential operators, Integral Equations Operator Theory 34 (1999) 439–457.
[17] D. Han, D.H. Larson, Frames, bases, and group representation, Mem. Amer. Math. Soc. 147 (697) (2000), x+94 pp.
[18] F. John, Extremum problems with inequalities as subsidiary conditions, in: Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publishers, Inc., New York, 1948, pp. 187–204.
[19] W.B. Johnson, H.P. Rosenthal, M. Zippin, On bases, finite dimensional decompositions, and weaker structures in Banach spaces, Israel J. Math. 9 (1971) 488–504.
[20] B.S. Kashin, The widths of certain finite dimensional sets and classes of smooth functions, Izv. Akad. Nauk SSSR Ser. Math. 41 (2) (1977) 334–351, 478.
[21] Y. Lyubarskii, R. Vershynin, Uncertainty principles and vector quantization, preprint, arXiv: math/0611343v1.
[22] B. Maurey, G. Pisier, Caractérisation d'une classe d'espaces de Banach par des propriétés de séries aléatoires vectorielles, C. R. Acad. Sci. Paris Sér. A-B 277 (1973).
[23] B. Maurey, G. Pisier, Séries de variables aléatoires vectorielles independantes et propriétés géométriques des espaces de Banach, Studia Math. 58 (1) (1976) 45–90 (in French).
[24] V.D. Milman, G. Pisier, Banach spaces with a weak cotype 2 property, Israel J. Math. 54 (2) (1986) 139–158.
[25] M.I. Ostrovskii, Minimal-volume shadows of cubes, J. Funct. Anal. 176 (2) (2000) 317–330.
[26] A. Pełczyński, Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with basis, Studia Math. 40 (1971) 239–242.
[27] G. Pisier, Un théorème sur les opérateurs linéaires entre espaces de Banach qui se factorisent par un espace de Hilbert, Ann. Sci. École Norm. Sup. (4) 13 (1) (1980) 23–43.
[28] G. Pisier, The Volume of Convex Bodies and Banach Space Geometry, Cambridge Tracts in Math., vol. 94, Cambridge University Press, Cambridge, 1989, xvi+250 pp.
[29] M. Rogalski, Sur le quotient volumique d'un espace de dimension finie, in: G. Choquet, M. Rogalski, J. Saint-Raymond (Eds.), Initiation Seminar on Analysis, 20th Year: 1980/1981, in: Publ. Math. Univ. Pierre et Marie Curie, vol. 46, Univ. Paris VI, Paris, 1981, Comm. No. C3, 31 pp.