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A NEW APPROACH TO THE RAMSEY-TYPE GAMES AND THE GOWERS DICHOTOMY IN F-SPACES

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We give a new approach to the Ramsey-type results of Gowers on block bases in Banach spaces and apply our results to prove the Gowers dichotomy in F-spaces.

1. Introduction

Our aim in this note is to establish the Gowers dichotomy [4] in a general F-space (complete metric linear space). We say that an F-space X is *hereditarily indecomposable* if it is impossible to find two *separated* infinite-dimensional closed subspaces V,W, i.e., such that $V\cap W=\{0\}$ and $V+W$ is closed (or equivalently that the natural projection from $V+W$ onto V is continuous). Our main result is that an F-space either contains an unconditional basic sequence or an infinite-dimensional HI subspace. In order to prove such a result we give a new and, we hope, interesting approach to the Gowers Ramsey-type result about block bases in a Banach space. We now state this result (terminology is explained in §2 and in [5], [6]):

Theorem 1.1 **([4], [5], [6]).** *Let X be a Banach space with a basis. Let ∂B_X denote the unit sphere of X, i.e., $\partial B_X=\{x\in X\colon \|x\|=1\}$. Let $\sigma\subseteq \Sigma_{<\infty}(\partial B_X)$. Let $\Theta=(\theta_i)_i$ be a sequence of positive numbers. If σ is large then there exists a block subspace Y of X such that σ_Θ is strategically large*

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for Y, where σ_Θ is the set of all finite block bases $\{u_1,\dots,u_n\}$ such that for some $\{v_1,\dots,v_n\}\in\sigma$ we have $\|u_i-v_i\|<\theta_i$.

In [5] and [6] the statement of Theorem 1.1 is announced for ∂B_X replaced by the unit ball except the origin, i.e., $B=B_X\setminus\{0\}=\{x\in X: 0<\|x\|\le 1\}$. There appears to be a slight problem in the non-normalized case in [5, page 805, line -9] and [6, page 1092, line -2], namely, it is used that the size of coefficients of a normalized vector with respect to a basic sequence of norm at most 1, is controlled from above by the basis constant. Theorem 1.1 (including the non-normalized case) follows from our Theorem 3.8.

Gowers also considers an infinite version of the same result (Theorem 4.1 of [5]):

Theorem 1.2 ([5], [6]). *Let X be a Banach space with a basis. Let $\sigma\subseteq\Sigma_\infty(\partial B_X)$. Let $\Theta=(\theta_i)_i$ be a sequence of positive numbers. If σ is analytic and large then there exists a block subspace Y of X such that σ_Θ is strategically large for Y, where σ_Θ is the set of all infinite block bases $\{u_1,\dots,u_n,\dots\}$ such that for some $\{v_1,\dots,v_n,\dots\}\in\sigma$ we have $\|u_i-v_i\|<\theta_i$.*

Other proofs of these results can be found in the work of Bagaria and López-Abad [1], [2]. Direct proofs of the dichotomy result without these theorems can be found in [13] and [3]; see also [14].

Our main objective is to prove Theorem 1.2 in a form that is suitable for our intended applications. We take a somewhat different viewpoint (see Theorem 4.4 below) by treating this theorem as a result about block bases in a countable dimensional space E with no topology assumed. We consider in fact only the intrinsic topology on E, i.e., the finest vector space topology. We then give a proof which is rather distinct from that given by Gowers, and we feel has some advantages. A benefit of this approach is that we are able to apply the result very easily to the setting of a general F-space.

In §5 we prove that the Gowers dichotomy extends to general F-spaces and discuss connections with similar (but easier) dichotomies for the existence of basic sequences. In the final section, §6 we prove the result of Gowers and Maurey [7] that on a complex HI-space every operator is the sum of a scalar and a strictly singular operator in the context of quasi-Banach spaces. This generalization is not entirely trivial and requires a few new tricks, although we broadly follow the same ideas as Gowers and Maurey.

2. Countable dimensional vector spaces

Let E be a real or complex vector space of countable algebraic dimension (this is usually denoted by c_{00} in the literature). There is a natural intrinsic topology $\mathcal{T}=\mathcal{T}_E$ on E defined as follows: *a set U is $\mathcal{T}$-open if $U\cap F$ is open relative to F for every finite-dimensional subspace F*. The topology $\mathcal{T}$ is a vector topology on E and is, indeed, the finest vector topology on E. It is known that $(E,\mathcal{T})$ is in fact locally convex. More precisely if $(e_j)_{j=1}^{\infty}$ is any fixed Hamel basis then the topology is induced by the family of norms

$$\left\|\sum_{j=1}^{m} a_j e_j\right\|_{\lambda} = \max_{1\le j\le m} \lambda_j |a_j|$$

where $\lambda=(\lambda_j)_{j=1}^{\infty}$ is any sequence of positive numbers. In the case when $\lambda_j=1$ for all j we denote the resulting norm by $\|\cdot\|_{\infty}$.

We will also be concerned with the product $E^{\mathbb{N}}$. On this there are two natural topologies: the *product topology* $\mathcal{T}_p$ and the *box topology*. The box topology $\mathcal{T}_{bx}$ is a topology which makes $E^{\mathbb{N}}$ a topological group but not a topological vector space. A base of neighborhoods of the origin for the box topology is given by sets of the form $\prod_{n=1}^{\infty} U_n$ where each U_n is a $\mathcal{T}$-neighborhood of zero in E. A base can also be given by sets of the form $\prod_{n=1}^{\infty}\{x\colon \|x\|_{\lambda}<\delta_n\}$ for some fixed norm $\|\cdot\|_{\lambda}$ and a sequence $\delta_n>0$. We observe the obvious fact that if V is an infinite-dimensional subspace of E then $\mathcal{T}_E|V=\mathcal{T}_V$ and $\mathcal{T}_{E,bx}|V^{\mathbb{N}}=\mathcal{T}_{V,bx}$.

Now let us suppose that E has a given fixed Hamel basis $(e_n)_{n=1}^{\infty}$. Let $E_n=[e_1,\dots,e_n]$ and $E^{(n)}=[e_{n+1},e_{n+2},\dots]$, where $[\dots]$ denotes the linear span. A sequence $(v_k)_{k=1}^{n}$ where $1\le n\le\infty$ is called a *block basis* of $(e_k)_{k=1}^{\infty}$ if each $v_k\neq 0$ and

$$v_k=\sum_{j=p_{k-1}+1}^{p_k} a_j e_j$$

for some increasing sequence $p_0=0<p_1<p_2<\cdots$. A subspace V of E is called a block subspace if V is the linear span of a block basis.

We let $\Sigma_{\infty}(E)$ be the subset of $E^{\mathbb{N}}$ consisting of all infinite block bases. For each $n\in\mathbb{N}$ we let $\Sigma_n(E)$ be the subset of $E^{\mathbb{N}}$ of all block bases of length n. We also let $\Sigma_0(E)$ be the one-point set with a single member $\emptyset$. Let $\Sigma_{<\infty}(E)$ denote the union of all $\Sigma_n(E)$ for $0\le n<\infty$. If A is a subset of E we denote by $\Sigma_n(A)$, etc., the subset of $\Sigma_n(E)$ with each element in A. In particular we will be interested in the sets

$$A_{\infty}=\{x\in E\colon\ 0<\|x\|_{\infty}\le 1\},\qquad S_{\infty}=\{x\in E\colon \|x\|_{\infty}=1\}.$$

Lemma 2.1. *Let $\|\cdot\|$ be any norm on E so that $(e_n)_{n=1}^{\infty}$ is a Schauder basis of the completion $\tilde{E}$ of $(E,\|\cdot\|)$. Then, on the space $\Sigma_{\infty}(E)$ the product topology $\mathcal{T}_p$ coincides with the product topology induced by $\|\cdot\|$. In particular $(\Sigma_{\infty},\mathcal{T}_p)$ is a Polish space.*

Proof. Let $\xi_n = (\xi_{n,k})_{k=1}^\infty$ be a sequence in $\Sigma_\infty(E)$ so that for some $\xi = (\xi_k)_{k=1}^\infty \in \Sigma_\infty(E)$, $\lim_{n\to\infty} \|\xi_{n,k} - \xi_k\| = 0$ for each k. Let us suppose $\xi_{n,k} \in [e_{p_{n,k-1}+1}, \dots, e_{p_{n,k}}]$ where $p_{n,0} < p_{n,1} < \cdots$ and that $\xi_k \in [e_{p_{k-1}+1}, \dots, e_{p_k}]$. Then it is clear that

$$\limsup_{n\to\infty} p_{n,k-1} \le p_k, \qquad k = 1, 2, \dots$$

using the fact that $(e_n)_{n=1}^\infty$ is a Schauder basis. It follows that each sequence $(\xi_{n,k})_{n=1}^\infty$ is contained in some fixed finite-dimensional space and so the convergence is also in $\mathcal{T}_p$.

For the product-norm topology it is also easy to see that $\Sigma_\infty(E)$ is a closed subset of $(\tilde{E} \setminus \{0\})^{\mathbb{N}}$ and hence is Polish. ∎

Let $\mathcal{B} = \mathcal{B}(E)$ be the collection of all infinite-dimensional block subspaces of $(e_k)_{k=1}^\infty$. If $V \in \mathcal{B}$ then V is the span of a block basis $(v_n)_{n=1}^\infty$ and we write $\mathcal{B}(V)$ for the collection of infinite-dimensional block subspaces of V with respect to $(v_n)_{n=1}^\infty$ (this is clearly independent of the choice of the block basis). We will use the notation $(v_1, \dots, v_r) \prec (u_1, \dots, u_s)$ to mean that $(v_1, \dots, v_r)$ is a block basis of $(u_1, \dots, u_s)$.

Let σ be a subset of $\Sigma_\infty(E)$. We shall say that σ is *large* if for every $V \in \mathcal{B}(E)$ we have $\sigma \cap \Sigma_\infty(V) \neq \emptyset$.

A *strategy* is a map $\Phi\colon \Sigma_{<\infty}(E) \times \mathcal{B}(E) \to \Sigma_{<\infty}(E)$ if for all $(u_1, \dots, u_n) \in \Sigma_n(E)$ we have $\Phi(u_1, u_2, \dots, u_n; V) = (u_1, \dots, u_n, u_{n+1})$ with $u_{n+1} \in V$.

If $(V_j)_{j=1}^\infty$ is a sequence of block subspaces then we will write

$$\Phi(u_1, \dots, u_n; V_1, \dots, V_m) = (u_1, \dots, u_{m+n})$$

and

$$\Phi(u_1, \dots, u_n; V_1, \dots, V_m, \dots) = (u_1, \dots, u_{m+n}, \dots)$$

where $u_{n+k} = \Phi(u_1, \dots, u_{n+k-1}; V_k)$ for $k \ge 1$. In the case when $n = 0$ we write $\Phi(V_1, \dots, V_m)$ or $\Phi(V_1, \dots, V_m, \dots)$ for $\Phi(\emptyset; V_1, \dots, V_m)$ or $\Phi(\emptyset; V_1, \dots, V_m, \dots)$.

A subset σ of $\Sigma_\infty(E)$ is called *strategically large* for $V \in \mathcal{B}(E)$ and $(u_1, \dots, u_n) \in \Sigma_{<\infty}(E)$ if there is a strategy Φ with the property that for every sequence $(V_j)_{j=1}^\infty$ with $V_j \subset V$ we have

$$\Phi(u_1, \dots, u_n; V_1, \dots, V_m, \dots) \in \sigma.$$

σ is strategically large for $V \in \mathcal{B}(E)$ if it is strategically large for $V \in \mathcal{B}(E)$ and $\emptyset$.

3. Functions on subsets of $\Sigma_{<\infty}(E)$

If V, W are subspaces of E let us write $V \subset_a W$ to mean that there exists a finite dimensional subspace F so that $V \subset W + F$.

Lemma 3.1 **(Stabilization Lemma).** *Let E be a countable dimensional space with fixed Hamel basis $(e_k)_{k=1}^{\infty}$. Let X be a separable topological space and suppose that, for each $V \in \mathcal{B}(E)$, $f_V\colon X \to \mathbb{R}$ is a continuous function. Suppose further that*

$$f_{V_1}(x) \geq f_{V_2}(x), \qquad x \in X$$

whenever $V_1 \subset_a V_2$. Then there is a block subspace W of E so that $f_V = f_W$ whenever $V \subset W$.

More generally suppose $(X_n)_{n=1}^{\infty}$ is a sequence of separable topological spaces and for each $V \in \mathcal{B}$ and $n \in \mathbb{N}$, $f_V^{(n)}\colon X_n \to \mathbb{R}$ is a continuous function. Suppose further that

$$f_{V_1}^{(n)}(x) \geq f_{V_2}^{(n)}(x), \qquad x \in X_n$$

whenever $V_1 \subset_a V_2$. Then there is a block subspace W of E so that $f_V^{(n)} = f_W^{(n)}$ whenever $V \subset_a W$ and $n \in \mathbb{N}$.

Proof. We prove the first part. We define block subspaces V_α for every countable ordinal α by transfinite induction, so that $\alpha \leq \beta \implies V_\beta \subset_a V_\alpha$. Set $V_1 = E$. For each α which is not a limit ordinal, say $\alpha = \beta + 1$ define $V_\alpha \subset V_\beta$ so that $f_{V_\alpha} \neq f_{V_\beta}$ if possible; otherwise let $V_\alpha = V_\beta$. If α is a limit ordinal then $\alpha = \sup_n \beta_n$ for some increasing sequence $(\beta_n)_{n=1}^{\infty}$ with $\beta_n < \alpha$. Thus $V_{\beta_m} \subset_a V_{\beta_n}$ if $m > n$. In this case we may by a diagonal argument find V_α so that $V_\alpha \subset_a V_{\beta_n}$ for every n (simply choose a block basis v_n with $v_n \in V_{\beta_1} \cap \cdots \cap V_{\beta_n}$). Now it follows that the functions f_{V_α} are increasing in α for $1 \leq \alpha < \omega_1$. If D is a countable dense set in X there must therefore exist a countable ordinal β so that

$$f_{V_\beta}(x) = f_{V_\alpha}(x), \qquad x \in D,\ \beta \leq \alpha.$$

Thus $f_{V_{\beta+1}} = f_{V_\beta}$ so that $W = V_\beta$ satisfies the conclusion.

The second part reduces to the first if we consider $X = \bigcup_{n=1}^{\infty} X_n$ topologized as a disjoint union and $f_V\colon X \to \mathbb{R}$ given by $f_V(x) = f_V^{(n)}(x)$ when $x \in X_n$. ∎

Consider a function $f\colon \Sigma_{<\infty}(A) \to [0,\infty)$ where $A = S_\infty$ or $A = A_\infty$. We shall say that f is *uniformly $\mathcal{T}_{bx}$-continuous* if given $\epsilon > 0$ there is a sequence $(U_n)_{n=1}^\infty$ of $\mathcal{T}$-neighborhoods of 0 such that if $(u_1,\dots,u_r),(v_1,\dots,v_r) \in \Sigma_{<\infty}(A)$ and $u_j - v_j \in U_j$ for $1 \le j \le r$ then

$$|f(u_1,\dots,u_r) - f(v_1,\dots,v_r)| < \epsilon.$$

In effect if we introduce maps $f^{[n]}$ on $\Sigma_\infty(A)$ by

$$f^{[n]}(u_1,\dots,u_k,\dots) = f(u_1,\dots,u_n)$$

this requires that the family of functions $(f^{[n]})_{n=1}^\infty$ is equi-uniformly continuous for the box topology $\mathcal{T}_{bx}$.

We will need a slightly weaker notion for maps $f\colon \Sigma_{<\infty}(A_\infty) \to [0,\infty)$. We will say that f is *admissible* if it is bounded and

(i) given $\epsilon > 0$, there is a sequence $(U_n)_{n=1}^\infty$ of $\mathcal{T}$-neighborhoods of 0 such that if $(u_1,\dots,u_r),(v_1,\dots,v_r) \in \Sigma_{<\infty}(S_\infty)$ and $u_j - v_j \in U_j$ for $1 \le j \le r$ then

$$|f(\lambda_1 u_1,\dots,\lambda_r u_r) - f(\lambda_1 v_1,\dots,\lambda_r v_r)| < \epsilon, \qquad (\lambda_1,\dots,\lambda_r) \in (0,1]^r;$$

and

(ii) given $\epsilon > 0$ and $(u_1,\dots,u_r) \in \Sigma_{<\infty}(S_\infty)$ there exists $\delta = \delta(u_1,\dots,u_r,\epsilon) > 0$ so that if $0 < \lambda_j, \mu_j \le 1$ for $1 \le j \le r$ and $\max_{1\le j\le r} |\lambda_j - \mu_j| \le \delta$ then

$$|f(\lambda_1 u_1,\dots,\lambda_r u_r, v_1,\dots,v_s) - f(\mu_1 u_1,\dots,\mu_r u_r, v_1,\dots,v_s)| < \epsilon,$$

whenever $(u_1,\dots,u_r,v_1,\dots,v_s) \in \Sigma_{<\infty}(A_\infty)$.

The following Lemma is easy and its proof is omitted:

Lemma 3.2. *(i) Suppose $f\colon \Sigma_{<\infty}(A_\infty) \to [0,\infty)$ is bounded and uniformly $\mathcal{T}_{bx}$-continuous; then f is admissible.*
(ii) Suppose $f\colon \Sigma_{<\infty}(S_\infty) \to [0,\infty)$ is uniformly $\mathcal{T}_{bx}$-continuous; then $g\colon \Sigma_{<\infty}(A_\infty) \to [0,\infty)$ is admissible where $g(u_1,\dots,u_n) = f(u_1/\|u_1\|_\infty, \dots, u_n/\|u_n\|_\infty)$.

Lemma 3.3. *If $f\colon \Sigma_{<\infty}(A_\infty) \to [0,\infty)$ is admissible then for each $m \in \mathbb{N}$ the map $F_m\colon (0,1]^m \times \Sigma_m(S_\infty) \to [0,\infty)$ defined by*

$$F(\lambda_1,\dots,\lambda_m,u_1,\dots,u_m) = f(\lambda_1 u_1,\dots,\lambda_m u_m)$$

is continuous when $\Sigma_m(S_\infty) \subset (E,\mathcal{T})^m$ is given the subset topology.

Proof. Suppose $\epsilon > 0$. We pick $\mathcal{T}$-neighborhoods of zero in E, $U_1, \ldots, U_m$ so that $u_j - v_j \in U_j$ for $1 \le j \le m$ implies that

$$|f(\lambda_1 v_1, \ldots, \lambda_m v_m) - f(\lambda_1 u_1, \ldots, \lambda_m u_m)| < \epsilon/2$$

for every $(\lambda_1, \ldots, \lambda_m) \in (0,1]^m$. If $(v_1, \ldots, v_m) \in \Sigma_m(E_n \cap S_\infty)$ we then pick $\delta = \delta(v_1, \ldots, v_m) > 0$ so that if $|\lambda_j - \mu_j| < \delta$ for $1 \le j \le m$ we have

$$|f(\lambda_1 v_1, \ldots, \lambda_m v_m) - f(\mu_1 v_1, \ldots, \mu_m v_m)| < \epsilon/2.$$

Combining gives

$$|f(\lambda_1 u_1, \ldots, \lambda_m u_m) - f(\mu_1 v_1, \ldots, \mu_m v_m)| < \epsilon$$

whenever $\max_{1 \le j \le m} |\lambda_j - \mu_j| < \delta$ and $u_j - v_j \in U_j$ for $1 \le j \le m$. Thus F is continuous at each point $(\mu_1, \ldots, \mu_m, v_1, \ldots, v_m)$. ∎

Suppose $f \colon \Sigma_{<\infty}(A_\infty) \to [0,\infty)$ is any admissible function. Let us adopt the convention that the function f takes the value $+\infty$ at any point of $E^{\mathbb{N}} \setminus \Sigma_{<\infty}(A_\infty)$. For any $V \in \mathcal{B}(E)$ define the function f'_V on $\Sigma_{<\infty}(A_\infty)$ by

$$f'_V(u_1, \ldots, u_n) = \lim_{m \to \infty} \inf\left\{f(u_1, \ldots, u_n, v_1, \ldots, v_s) \colon v_1, \ldots, v_s \in V \cap E^{(m)},\ s \ge 1\right\}.$$

Note that $V \subset_a W$ implies that $f'_V \ge f'_W$.

The following is more or less immediate:

Lemma 3.4. *If $f \colon \Sigma_{<\infty}(A_\infty) \to [0,\infty)$ is admissible, then each of the functions $f'_V \colon \Sigma_{<\infty}(A_\infty) \to [0,\infty)$ is admissible.*

Lemma 3.5. *If $\mathcal{F}$ is a countable family of admissible functions, then there exists $V \in \mathcal{B}(E)$ so that for every $W \in \mathcal{B}(V)$ and every $f \in \mathcal{F}$ we have $f'_W = f'_V$.*

Proof. For $W \in \mathcal{B}(E)$ and $m < n$ define $g_{m,n,W} \colon (0,1]^m \times \Sigma_m(S_\infty \cap E^n) \to \mathbb{R}$ by

$$g_{m,n,W}(\lambda_1, \ldots, \lambda_m, u_1, \ldots, u_m) = f'_W(\lambda_1 u_1, \ldots, \lambda_m u_m).$$

Thus $g_{m,n,W}$ is continuous by Lemma 3.3. Since $(0,1]^m \times \Sigma_m(S_\infty \cap E^n)$ is separable for each m, n, we can apply the Stabilization Lemma 3.1. ∎

We can thus assume, under the hypotheses of the Lemma (by passing to a block subspace), that f has the property that $f'_V = f'_E$ for all block subspaces V. If this happens we shall say that f is *stable* and we write f' for f'_E. Note that f' is admissible.

Proposition 3.6. *Let f be a stable admissible function. Suppose $(u_1,\dots,u_r)\in\Sigma_{<\infty}(A_\infty)$ and V is a block subspace. Then for any $\epsilon>0$ there exists $\xi\in V\setminus\{0\}$ so that either:*

(a) $f(u_1,\dots,u_r,\xi)<f'(u_1,\dots,u_r)+\epsilon$; or
(b) $f'(u_1,\dots,u_r,\xi)<f'(u_1,\dots,u_r)+\epsilon$.

Proof. Let us assume that $(u_1,\dots,u_r)\in\Sigma_r(E)$. Let us further assume that V is a block subspace so that for any $\xi\in V$ we have

$$\begin{aligned} f(u_1,\dots,u_r,\xi) &\geq f'(u_1,\dots,u_r)+\epsilon,\\ f'(u_1,\dots,u_r,\xi) &\geq f'(u_1,\dots,u_r)+\epsilon. \end{aligned} \tag{3.1}$$

Let $(v_j)_{j=1}^\infty$ be a block basis which is a basis of V. We choose an increasing sequence of integers $(q_k)_{k=1}^\infty$ as follows. Let $q_1=1$. Assume $q_1,\dots,q_k$ have been chosen. Let m_0 be the smallest integer so that $v_{q_k}\in E_{m_0}$. Then for every $\xi\in S_\infty\cap[v_{q_1},\dots,v_{q_k}]$ (3.1) holds. We may pick a neighborhood U of 0 in $(E,\mathcal{T})$ so that

$$|f(u_1,\dots,u_r,\lambda\xi,w_1,\dots,w_s)-f(u_1,\dots,u_r,\lambda\eta,w_1,\dots,w_s)|<\epsilon/8$$

when $\xi-\eta\in U$, $(w_1,\dots,w_s)\in\Sigma_{<\infty}(A_\infty\cap E^{(m_0)})$ and $0<\lambda\leq 1$. By compactness there is a finite subset $(\xi_1,\dots,\xi_t)$ of $S_\infty\cap[v_{q_1},\dots,v_{q_k}]$ such that $\eta\in S_\infty\cap[v_{q_1},\dots,v_{q_k}]$ implies $\eta-\xi_j\in U$ for some j. Now pick an integer N large enough so that $|\lambda-\mu|<1/N$ implies

$$|f(u_1,\dots,u_r,\lambda\xi_j,w_1,\dots,w_s)-f(u_1,\dots,u_r,\mu\xi_j,w_1,\dots,w_s)|<\epsilon/8$$

whenever $(w_1,\dots,w_s)\in\Sigma_{<\infty}(E^{(m_0)})$. Now by our assumptions we can pick $m\geq m_0$ so that

$$f\big(u_1,\dots,u_r,\tfrac{k}{N}\xi_j,w_1,\dots,w_s\big)>f'(u_1,\dots,u_r)+\tfrac{3}{4}\epsilon$$

whenever $1\leq j\leq t$, $1\leq k\leq N$ and $(w_1,\dots,w_s)\in\Sigma_{<\infty}(E^{(m)})$. Hence

$$f(u_1,\dots,u_r,\lambda\xi_j,w_1,\dots,w_s)>f'(u_1,\dots,u_r)+\tfrac{5}{8}\epsilon$$

whenever $1\leq j\leq t$, $0<\lambda\leq 1$ and $(w_1,\dots,w_s)\in\Sigma_{<\infty}(E^{(m)})$, and thus

$$f(u_1,\dots,u_r,\lambda\xi,w_1,\dots,w_s)>f'(u_1,\dots,u_r)+\tfrac{1}{2}\epsilon \tag{3.2}$$

whenever $0<\lambda\leq 1$ and $\xi\in S_\infty\cap[v_{q_1},\dots,v_{q_k}]$.

Then we pick $q_{k+1}>q_k$ so that $v_{q_{k+1}}\in E^{(m)}$. This completes the inductive construction. Now let $W=[v_{q_1},v_{q_2},\dots]$. There exists $(w_1,\dots,w_s)\in\Sigma_s(W)$ (where $s>0$) so that $f(u_1,\dots,u_r,w_1,\dots,w_s)<f'(u_1,\dots,u_r)+\epsilon/2$.

If $s=1$ then $\xi=w_1$ contradicts (3.1). If $s>1$ let $\lambda\xi=w_1=\sum_{j=1}^l a_jv_{q_j}$ where $a_l\neq 0$. Then by the selection of q_{l+1} and (3.2) we see that $f(u_1,\dots,u_r,w_1,\dots,w_s)>f'(u_1,\dots,u_r)+\epsilon/2$, a contradiction. ∎

Lemma 3.7. *If* $f\colon \Sigma_{<\infty}(A_\infty) \to [0,\infty)$ *is admissible then the function* $g\colon \Sigma_{<\infty}(A_\infty) \to [0,\infty)$ *given by* $g(u_1,\dots,u_r)=1$ *if* $r=0$ *or* $r=1$ *and*

$$g(u_1,\dots,u_r) = \inf\{f(v_1,\dots,v_s)\colon (v_1,\dots,v_s) \prec (u_1,\dots,u_r),\ 1 \le s < r\}$$

(if $r>1$*) is admissible.*

Proof. Note that g satisfies $g(\lambda_1 u_1,\dots,\lambda_m u_m) = g(u_1,\dots,u_m)$ if $(u_1,\dots,u_m) \in \Sigma_{<\infty}(S_\infty)$ and $(\lambda_1,\dots,\lambda_m) \in (0,1]^m$. Hence it suffices to show that g is uniformly $\mathcal{T}_{bx}$-continuous on $\Sigma_{<\infty}(S_\infty)$. Note that, if for all $(u_1,\dots,u_m) \in \Sigma_{<\infty}(S_\infty)$ we define

$$h(u_1,\dots,u_m) = \inf\{f(\lambda_1 u_1,\dots,\lambda_m u_m)\colon (\lambda_1,\dots,\lambda_m) \in (0,1]^m\},$$

then h is uniformly $\mathcal{T}_{bx}$-continuous and

$$g(u_1,\dots,u_r) = \inf\{h(v_1,\dots,v_s)\colon (v_1,\dots,v_s) \prec (u_1,\dots,u_r),\ 1 \le s < r\}, \qquad r > 1.$$

Suppose $\epsilon > 0$. Then there is a sequence $(U_n)_{n=1}^\infty$ of $\mathcal{T}$-neighborhoods of zero so that if $(u_1,\dots,u_n), (v_1,\dots,v_n) \in \Sigma_{<\infty}(S_\infty)$ and $u_j - v_j \in U_j$ for $1 \le j \le n$ then

$$|h(u_1,\dots,u_n) - h(v_1,\dots,v_n)| < \epsilon.$$

Pick a sequence of circled neighborhoods of zero, $(U'_n)_{n=1}^\infty$, so that $U'_n + U'_{n+1} + \dots + U'_{n+k} \subset U_n$ whenever $k \ge n$. Then suppose $(u_1,\dots,u_n), (v_1,\dots,v_n) \in \Sigma_{<\infty}(S_\infty)$ with $u_j - v_j \in U'_j$ for $1 \le j \le n$. Assume $\eta > 0$ and pick $(x_1,\dots,x_r) \prec (u_1,\dots,u_n)$ with $r < n$ so that $h(x_1,\dots,x_r) < g(u_1,\dots,u_n) + \eta$. If $x_j = \sum_{i=k+1}^l a_i u_i$ let $y_j = \sum_{i=k+1}^l a_i v_i$. Then $|a_i| \le 1$ and so $x_j - y_j \in U'_{k+1} + \dots + U'_l \subset U_j$ since $j \le k+1$. Hence

$$h(y_1,\dots,y_r) < g(u_1,\dots,u_n) + \epsilon + \eta$$

and so

$$g(v_1,\dots,v_n) \le g(u_1,\dots,u_n) + \epsilon.$$

By symmetry

$$|g(v_1,\dots,v_n) - g(u_1,\dots,u_n)| \le \epsilon$$

and hence g is uniformly $\mathcal{T}_{bx}$-continuous. ∎

We shall say that a strategy Φ is (ϵ, V)-effective for f where $\epsilon > 0$ and $V \in \mathcal{B}(E)$ if for every sequence of block subspaces $V_j \subset V$ there exists $n \in \mathbb{N}$ so that

$$f(\Phi(\emptyset, V_1,\dots,V_n)) < \sup_{W \in \mathcal{B}(E)} f'_W(\emptyset) + \epsilon.$$

If $(u_1,\dots,u_r)\in\Sigma_{<\infty}(E)$ we shall say that a strategy Φ is $(\epsilon,u_1,\dots,u_r,V)$-effective for f where $\epsilon>0$ and $V\in\mathcal{B}(E)$ if for every sequence of block subspaces $V_j\subset V$ there exists $n\in\mathbb{N}$ so that

$$f(\Phi(u_1,\dots,u_r,V_1,\dots,V_n)) < \sup_{W\in\mathcal{B}(E)} f'_W(u_1,\dots,u_r)+\epsilon.$$

Theorem 3.8. *Suppose $f\colon \Sigma_{<\infty}(A_\infty)\to[0,\infty)$ is admissible. Then, given $\epsilon>0$ there is a block subspace V and a strategy Φ which is (ϵ,V)-effective for f.*

Proof. Before presenting the details of the proof we outline it: First we assume without loss of generality that $f'_V(\emptyset)=0$ for all $V\in\mathcal{B}(E)$. Then we add a penalty function to f to define a function h. We pass to a block subspace to stabilize h to h'. The penalty function makes sure that for every $W=[u_1,u_2,\dots]\in\Sigma_\infty(A_\infty)$ there exists an integer r such that $h'(u_1,\dots,u_r)$ is large. Then for some sequence (ϵ_j) of small positive numbers we inductively use Proposition 3.6 to define the strategy, so that at each step, either h or h' is controlled by the value of h' at the previous step. Because of the penalty function, it is impossible that always h' is controlled. So at some step h is controlled. The first time that this happens gives you the result!

Now let's go over the proof again, slowly this time, and see the details: We assume (after stabilization) that $f'_V(\emptyset)=0$ for all $V\in\mathcal{B}(E)$; indeed if $a=\sup_{V\in\mathcal{B}(E)} f'_V(\emptyset)$ then replace f by $\max(f-a,0)$. We consider the admissible function

$$h(u_1,\dots,u_r)=f(u_1,\dots,u_r)+2(\epsilon-2g(u_1,\dots,u_r))_+,$$

where g is the function defined in Lemma 3.7. By Lemma 3.4 we can pass to a block subspace where h is stable; so let us assume h is stable on E.

We first claim that if $(u_1,\dots,u_n,\dots)\in\Sigma_\infty(A_\infty)$ then there exists n so that $h'(u_1,\dots,u_n)>\epsilon$. Indeed, let $W=[u_1,\dots,u_n,\dots]$. Then, since $f'_W(\emptyset)=0$, there exists $(v_1,\dots,v_s)\in\Sigma_{<\infty}(W)$ with $f(v_1,\dots,v_s)<\epsilon/4$. Then we may find $r>s$ so that $(v_1,\dots,v_s)\prec(u_1,\dots,u_r)$. Hence for any $(x_1,\dots,x_t)$ such that $(u_1,\dots,u_r,x_1,\dots,x_t)\in\Sigma_{<\infty}(A_\infty)$ we have

$$h(u_1,\dots,u_r,x_1,\dots,x_t)\ge 2(\epsilon-2f(v_1,\dots,v_s))$$

so that

$$h'(u_1,\dots,u_r)\ge 2(\epsilon-2f(v_1,\dots,v_s))>\epsilon,$$

which proves the claim.

On the other hand, given any block subspace V, there exists a minimal $s\ge 1$ so that we can find $(v_1,\dots,v_s)\in\Sigma_{<\infty}(V)$ with $f(v_1,\dots,v_s)<\epsilon/2$. Thus $g(v_1,\dots,v_s)\ge\epsilon/2$ which implies $h(v_1,\dots,v_s)<\epsilon/2$. Hence $h'(\emptyset)<\epsilon/2$.

We now use a strategy for h indicated by Proposition 3.6. Suppose $\epsilon_j > 0$ for each $j \geq 0$ and $\sum \epsilon_r < \epsilon/2$. Given $(u_1, \ldots, u_r) \in \Sigma_{<\infty}(E)$ and $V \in \mathcal{B}(E)$ we define $\Phi(u_1, \ldots, u_r, V)$ to be $(u_1, \ldots, u_r, \xi)$ where $\xi \in V \setminus \{0\}$ is chosen so that $h'(u_1, \ldots, u_r, \xi) < h'(u_1, \ldots, u_r) + \epsilon_r$ or $h(u_1, \ldots, u_r, \xi) < h'(u_1, \ldots, u_r) + \epsilon_r$. Let $(V_j)_{j=1}^{\infty}$ be a sequence in $\mathcal{B}(E)$ and let $(u_1, \ldots, u_r, \ldots) = \Phi(\emptyset; V_1, \ldots)$. Then, by our previous claim there exists a first $n \geq 1$ so that $h'(u_1, \ldots, u_n) > h'(u_1, \ldots, u_{n-1}) + \epsilon_{n-1}$. Hence

$$h(u_1, \ldots, u_n) < h'(u_1, \ldots, u_{n-1}) + \epsilon_{n-1} < h'(\emptyset) + \sum_{j=0}^{n-1} \epsilon_j < \epsilon. \qquad \blacksquare$$

We need an obvious extension of this result.

Theorem 3.9. *Suppose $f \colon \Sigma_{<\infty}(A_\infty) \to [0, \infty)$ is admissible. Then there is a block subspace V such that for each $(u_1, \ldots, u_r) \in \Sigma_{<\infty}(A_\infty)$ there is a strategy $\Phi_{u_1, \ldots, u_r}$ which is $(\epsilon, u_1, \ldots, u_r, V)$-effective for f.*

Proof. For each $(u_1, \ldots, u_r)$, it is easy to produce a block subspace $V_{u_1, \ldots, u_r}$ and device a strategy $\Psi_{u_1, \ldots, u_r}$ which is $(\epsilon/2, u_1, \ldots, u_r, V_{u_1, \ldots, u_r})$-effective for f. Indeed, suppose $u_r \in E_m$; define

$$f_1(v_1, \ldots, v_s) = f(u_1, \ldots, u_r, v_1, \ldots, v_s), \qquad (v_1, \ldots, v_s) \in \Sigma_{<\infty}\big(A_\infty \cap E^{(m)}\big)$$

and apply the preceding theorem to f_1. ($\Psi_{u_1, \ldots, u_r}$ has to be defined in some arbitrary fashion for $(w_1, \ldots, w_s)$ which do not have $(u_1, \ldots, u_r)$ as an initial segment.) Furthermore it can be seen that for each block subspace W we can choose $V_{u_1, \ldots, u_r} \subset W$.

To obtain a single block subspace V we first construct a dense countable subset $D_{m,r}$ in each $\Sigma_r(E_m \cap \mathcal{A}_\infty)$. We arrange the elements of $D = \bigcup_{m,r} D_{m,r}$ as a sequence and hence find a descending sequence of subspaces (V_n) so that the strategy $\Phi_{u_1, \ldots, u_r}$ is $(\epsilon/2, u_1, \ldots, u_r, V_n)$-effective for f when $(u_1, \ldots, u_r)$ is the nth member of D. If we select V to be block subspace so that $V \subset V_n + F_n$ for some finite-dimensional F_n for each n, then (via a simple modification) each $\Phi_{u_1, \ldots, u_r}$ is $(\epsilon/2, u_1, \ldots, u_r, V)$-effective for f. Finally we observe that if $(v_1, \ldots, v_r) \in \Sigma_r(E_m)$ is arbitrary and we then choose $(u_1, \ldots, u_r) \in D$ close enough, we can define a strategy by

$$\Phi_{v_1, \ldots, v_r}(v_1, \ldots, v_r, w_1, \ldots, w_s) = \Psi_{u_1, \ldots, u_r}(u_1, \ldots, u_r, w_1, \ldots, w_s)$$

(and arbitrarily otherwise) then we will have that $\Phi_{v_1, \ldots, v_n}$ is $(\epsilon, v_1, \ldots, v_r, V)$-effective for f. $\blacksquare$

4. The infinite case

We now turn to the infinite case. Suppose $f\colon \Sigma_\infty(A_\infty)\to[0,\infty)$ is a bounded uniformly $\mathcal{T}_{bx}$-continuous function. We may define $f'_V\colon \Sigma_{<\infty}(A_\infty)\to[0,\infty)$ in a precisely analogous way. As before we adopt the convention that $f=+\infty$ on $E^{\mathbb{N}}\setminus\Sigma_\infty(A_\infty)$. Let

$$f'_V(u_1,\dots,u_r)= \lim_{m\to\infty}\inf\{f(u_1,\dots,u_r,v_1,\dots,v_s,\dots)\colon v_j\in V\cap E^{(m)},\ j=1,2,\dots\}.$$

It is clear that the functions $\{f'_V\colon V\in\mathcal{B}(E)\}$ are equi-uniformly $\mathcal{T}_{bx}$-continuous.

Proceeding in the same manner as before we can show:

Proposition 4.1. *If $f_n\colon \Sigma_\infty(A_\infty)\to\mathbb{R}$ is any countable family of bounded $\mathcal{T}_{bx}$ uniformly continuous functions, there is a block subspace V of E so that $f'_W=f'_V$ whenever $W\in\mathcal{B}(V)$.*

We shall say that f is *stable* if $f'_E=f'_V$ for every $V\in\mathcal{B}(E)$.

Lemma 4.2. *Let $f_n\colon \Sigma_\infty(A_\infty)\to[0,\infty)$ be a sequence of bounded uniformly $\mathcal{T}_{bx}$-continuous functions and suppose $f=\inf f_n$ is also $\mathcal{T}_{bx}$-uniformly continuous. Assume that each f_n and f are stable. Let $h\colon \Sigma_{<\infty}(A_\infty)=\inf_n f'_n$. Then for every $V\in\mathcal{B}(E)$ we have $h'_V\le f'$.*

Proof. Let $V\in\mathcal{B}(E)$. Let us assume $h'_V(u_1,\dots,u_r)>\lambda>f'(u_1,\dots,u_r)$ for some $(u_1,\dots,u_r)\in\Sigma_{<\infty}(A_\infty\cap V)$. Then there exists m so that if $(v_1,\dots,v_s)\in\Sigma_{<\infty}(E^{(m)}\cap V)$ with $s\ge1$ we have

$$h(u_1,\dots,u_r,v_1,\dots,v_s)>\lambda.$$

Thus

$$f'_n(u_1,\dots,u_r,v_1,\dots,v_s)>\lambda,\qquad n=1,2\dots.$$

Let us pick $w_1\in A_\infty\cap E^{(m)}\cap V$. We will construct a sequence $(w_n)_{n=1}^\infty$ in V by induction. Suppose $(w_1,\dots,w_n)$ have been selected and let $W_n=[w_1,\dots,w_n]$. Then by compactness we can find p so that $W_n\subset E_p$ and if $1\le k\le n$ and $(v_1,\dots,v_k)\in\Sigma_k(W_n\cap A_\infty)$ then

$$f_k(u_1,\dots,u_r,v_1,\dots,v_k,x_1,x_2,\dots)>\lambda$$

for all choices of $(x_j)_{j=1}^\infty$ in $E^{(p)}$. Pick $w_{n+1}\in E^{(p)}\cap V$.

Now let $W=[w_j]_{j=1}^{\infty}$. Our construction guarantees that for every n it is true that

$$f_n(u_1,\ldots,u_r,x_1,x_2,\ldots)>\lambda, \qquad x_j\in W,\ j=1,2,\ldots.$$

Indeed we have $(x_1,\ldots,x_n)\in\Sigma_n([w_1,\ldots,w_m])$ where $m\ge n$ and so this follows from our inductive construction. Thus

$$f'(u_1,\ldots,u_r)\ge\lambda,$$

contradicting our initial hypothesis. ∎

We now use the space $\mathbb{N}^{\mathbb{N}}$ with the usual product topology; this can be regarded as the space of all infinite words of the natural numbers. We will write $\mathbb{N}^{<\infty}=\bigcup_{n=0}^{\infty}\mathbb{N}^k$ which is the space of all finite words of the natural numbers, including the empty word. We will use $(n_1,\ldots,n_k)$ or $(n_1,n_2,\ldots)$ to denote a typical member of $\mathbb{N}^{<\infty}$ or $\mathbb{N}^{\mathbb{N}}$ respectively.

Theorem 4.3. *Suppose $F\colon\mathbb{N}^{\mathbb{N}}\times\Sigma_\infty(A_\infty)\to[0,\infty)$ is a bounded map. Define $f_{n_1,n_2,\ldots}\colon\Sigma_\infty(A_\infty)\to[0,\infty)$ by*

$$f_{n_1,n_2,\ldots}(u_1,u_2,\ldots)=F(n_1,n_2,\ldots;u_1,u_2,\ldots).$$

Suppose

(i) the maps $\{f_{n_1,n_2,\ldots}\colon(n_1,n_2,\ldots)\in\mathbb{N}^{\mathbb{N}}\}$are equi-uniformly $\mathcal{T}_{bx}$-continuous;
(ii) the map $F\colon\mathbb{N}^{\mathbb{N}}\times\Sigma_\infty(A_\infty)\to[0,\infty)$ is lower semi-continuous for the product topology on $\mathbb{N}^{\mathbb{N}}\times(\Sigma_\infty(A_\infty),\mathcal{T}_p)$.

Let

$$f(u_1,u_2,\ldots)=\inf_{(n_1,n_2,\ldots)\in\mathbb{N}^{\mathbb{N}}}F(n_1,n_2,\ldots;u_1,u_2,\ldots).$$

If $f'_V(\emptyset)=0$ for every block subspace V, then given $\epsilon>0$ there is a block subspace V so that $\{f<\epsilon\}$ is V-strategically large.

Proof. For each $(n_1,\ldots,n_k)\in\mathbb{N}^{<\infty}$ we define

$$f_{n_1,n_2,\ldots,n_k}(u_1,u_2,\ldots)=\inf_{m_1,m_2,\ldots}F(n_1,\ldots,n_k,m_1,m_2,\ldots;u_1,u_2,\ldots).$$

The family $f_{n_1,n_2,\ldots,n_k}$ is $\mathcal{T}_{bx}$-equi-uniformly continuous. By passing to a block subspace we can assume that each $f_{n_1,n_2,\ldots,n_k}$ is stable. Of course the family $f'_{n_1,\ldots,n_k}$ is also $\mathcal{T}_{bx}$-equi-uniformly continuous. Let

$$h_{n_1,\ldots,n_k}(u_1,\ldots,u_r)=\inf_{m\in\mathbb{N}}f'_{n_1,\ldots,n_k,m}(u_1,\ldots,u_r), \quad (u_1,\ldots,u_r)\in\Sigma_{<\infty}(A_\infty).$$

This family is also equi-uniformly $\mathcal{T}_{bx}$-continuous. Passing to a further block subspace we can suppose that this family is also stable.

By Lemma 4.2 we have that $h'_{n_1,\dots,n_k} \le f'_{n_1,\dots,n_k}$.

Let us choose a sequence $(\epsilon_r)_{r=0}^{\infty}$ with $\sum \epsilon_r = \epsilon' < \epsilon$. Again, by the proof of Theorem 3.8, and by exploiting the countability of the family $h_{n_1,\dots,n_k}$ we can pass to a further block subspace and, by relabelling as E, suppose that for each $(n_1,\dots,n_k) \in \mathbb{N}^{<\infty}$ and $(u_1,\dots,u_r) \in \Sigma_{<\infty}(A_\infty)$ there is a strategy $\Phi_{n_1,\dots,n_k,u_1,\dots,u_r}$ with the property that if $(V_j)_{j=1}^{\infty}$ is any sequence of subspaces then for some $p \ge 1$,

$$h_{n_1,\dots,n_k}\Phi_{n_1,\dots,n_k,u_1,\dots,u_r}(u_1,\dots,u_r,V_1,\dots,V_p) < h'_{n_1,\dots,n_k}(u_1,\dots,u_r) + \epsilon_k.$$

We will now define a strategy Ψ. To do this we first define maps $\theta\colon \Sigma_{<\infty}(A_\infty) \to \mathbb{N}^{<\infty}$ and $\varphi\colon \Sigma_{<\infty}(A_\infty) \to \mathbb{N}$ such that $\varphi(u_1,\dots,u_r) \le r$. This is done inductively on the length of $(u_1,\dots,u_r)$. We define $\theta(\emptyset) = \emptyset$ and $\varphi(\emptyset) = 0$. Suppose that θ and φ have been defined for all ranks up to r and consider $(u_1,\dots,u_{r+1})$.

Let $\theta(u_1,\dots,u_r) = (n_1,\dots,n_k)$ and $\varphi(u_1,\dots,u_r) = s$. If

$$h_{n_1,\dots,n_k}(u_1,\dots,u_{r+1}) < h'_{n_1,\dots,n_k}(u_1,\dots,u_s) + \epsilon_k$$

then we can choose $m \in \mathbb{N}$ so that

$$f'_{n_1,\dots,n_k,m}(u_1,\dots,u_{r+1}) < h'_{n_1,\dots,n_k}(u_1,\dots,u_s) + \epsilon_k.$$

Let $\theta(u_1,\dots,u_{r+1}) = (n_1,n_2,\dots,n_k,m)$ and $\varphi(u_1,\dots,u_{r+1}) = r+1$.

Otherwise we simply put $\theta(u_1,\dots,u_{r+1}) = (n_1,\dots,n_k)$ and $\varphi(u_1,\dots,u_{r+1}) = s$.

To define Ψ we set

$$\Psi(u_1,\dots,u_r,V) = \Phi_{n_1,\dots,n_k,u_1,\dots,u_s}(u_1,\dots,u_r,V)$$

where $(n_1,\dots,n_k) = \theta(u_1,\dots,u_r)$ and $s = \varphi(u_1,\dots,u_r)$.

Finally, we must show that if $(V_j)_{j=1}^{\infty}$ is any sequence of subspaces the sequence $(u_1,u_2,\dots) = \Psi(\emptyset,V_1,V_2,\dots)$ is in $\{f < \epsilon\}$.

Let $k(r)$ be the length of $\theta(u_1,\dots,u_r)$. Then $k(r) \le r$ for all r. Suppose $k(r)$ remains bounded. Then there exists s so that $\varphi(u_1,\dots,u_r) = s$ for all $r \ge s$ and $\theta(u_1,\dots,u_k) = (n_1,\dots,n_t)$ for some fixed $(n_1,n_2,\dots,n_t) \in N^{<\infty}$ when $k \ge s$. Thus

$$(u_1,\dots,u_r) = \Phi_{n_1,\dots,n_t,u_1,\dots,u_s}(u_1,\dots,u_s,V_{s+1},\dots,V_r), \qquad r \ge s.$$

It follows that for some $r > s$ we have

$$h_{n_1,\dots,n_t}(u_1,\dots,u_r) < h'_{n_1,\dots,n_t}(u_1,\dots,u_s) + \epsilon_t$$

which implies that $\varphi(r)=r$ which is a contradiction. Hence $k(r)\uparrow\infty$. Let r_j be the first natural number at which $k(r)=j$. Then there exists $(n_1,n_2,\ldots)\in\mathbb{N}^{\mathbb{N}}$ so that

$$\theta(u_1,\ldots,u_{r_j})=(n_1,\ldots,n_j).$$

By construction

$$f'_{n_1}(u_1,\ldots,u_{r_1})<h'(\emptyset)+\epsilon_0$$

and then for $j\geq 1$,

$$f'_{n_1,\ldots,n_{j+1}}(u_1,\ldots,u_{r_{j+1}})<h'_{n_1,\ldots,n_j}(u_1,\ldots,u_{r_j})+\epsilon_j.$$

Since $h'_{n_1,\ldots,n_j}\leq f'_{n_1,\ldots,n_j}$ we conclude that

$$f'_{n_1,\ldots,n_j}(u_1,\ldots,u_{k_j})<\epsilon'$$

for all j. But this implies the existence of $(n_{j,1},n_{j,2},\ldots)\in\mathbb{N}^{\mathbb{N}}$ and $(u_{j,1},u_{j,2},\ldots)\in\Sigma_\infty(A_\infty)$ so that $n_{j,i}=n_i$ for $i\leq j$ and $u_{j,i}=u_i$ for $i\leq k_j$ and

$$F(n_{j,1},n_{j,2},\ldots;u_{j,1},u_{j,2},\ldots)<\epsilon',\qquad j=1,2,\ldots.$$

Finally we invoke lower semi-continuity:

$$F(n_1,n_2,\ldots;u_1,u_2,\ldots)<\epsilon$$

and so $f(u_1,u_2,\ldots)<\epsilon$. ∎

We now recall (Lemma 2.1) that $\Sigma_\infty(E)$ is a Polish space for the topology $\mathcal{T}_p$. Thus every Borel set is analytic (i.e., a continuous image of $\mathbb{N}^{\mathbb{N}}$).

Theorem 4.4. *Let σ be a large subset of $\Sigma_\infty(E)$. Suppose:*

(a) There is a sequence of absolutely convex sets C_n such that $C_n\cap F$ is compact for all finite-dimensional subspaces F and $\sigma\subset\prod_{n=1}^\infty C_n$.
(b) σ is analytic as a subset of $(\Sigma_\infty(E),\mathcal{T}_p)$.

Let ρ_n be any sequence of F-norms on E and define for $u=(u_1,u_2,\ldots)$, $v=(v_1,v_2,\ldots)\in\Sigma_\infty(E)$

$$d(u,v)=\sum_{j=1}^\infty\rho_j(u_j-v_j).$$

Let $\sigma_\epsilon=\{u=(u_j)_{j=0}^\infty\colon d(u,\sigma)=\inf_{v\in\sigma}d(u,v)<\epsilon\}$. Then for every $\epsilon>0$ there is a block subspace V so that σ_ϵ is strategically large for V.

Proof. We start by reducing this to the case when $C_n=\{x\colon \|x\|_\infty\le 1\}$. To do this first observe that each C_n is $\mathcal{T}$-closed. Since σ is large the linear space generated by C_n is of finite codimension; if E_n is a complementary space we can replace C_n by the bigger set C_n+K_n where K_n is a compact absolutely convex neighborhood of the origin in E_n. So we can suppose C_n is absorbent and hence generates a norm $\|\cdot\|_n$ on E. By induction, we can find a sequence of positive numbers δ_n so that $\|x\|=\sum_{n=1}^\infty \delta_n\|x\|_n<\infty$ for all $x\in E$. Thus we can assume that each $C_n=\{x\colon \|x\|\le M_n\}$ for a single norm $\|\cdot\|$.

We can now pass to block basis which is a normalized basic sequence in the completion of $(E,\|\cdot\|)$. Intersecting σ with $\Sigma_\infty(V)$ for a block subspace gives again an analytic set since $\Sigma_\infty(V)$ is closed; thus we can relabel so that the block subspace is already E. It now follows that each C_n is included in a set $\{x\colon \|x\|_\infty\le M_n'\}$ where M_n' is some sequence of positive numbers. Finally we put $\sigma'=\{(u_1,u_2,\dots)\colon (M_1'u_1,M_2'u_2,\dots)\in\sigma\}$ and note that $\sigma'\subset\Sigma_\infty(A_\infty)$. Clearly it is enough to prove the result for σ' with ρ_j replaced by $\rho_j'(x)=\rho_j(M_j'x)$.

We therefore assume that $\sigma\subset\Sigma_\infty(A_\infty)$.

Now there is a continuous surjective map $g\colon \mathbb{N}^\mathbb{N}\to\sigma$ for the $\mathcal{T}_p$-topology. We will define

$$F\colon \mathbb{N}^\mathbb{N}\times\Sigma_\infty(A_\infty)\to[0,\infty)$$

by

$$F(n_1,n_2,\dots;u_1,u_2,\dots)=\min\big(1,d((u_1,u_2,\dots),g(n_1,n_2,\dots))\big).$$

It is clear that the family $f_{n_1,n_2,\dots}$ given by

$$f_{n_1,n_2,\dots}(u_1,u_2,\dots)=F(n_1,n_2,\dots;u_1,u_2,\dots)$$

is equi-uniformly $\mathcal{T}_{bx}$-continuous. It is also clear that F is lower semicontinuous for the $\mathcal{T}_p$-topology in the second factor.

The result now follows directly from Theorem 4.3. ∎

5. Applications to F-spaces

Recall that an F-space is a complete metric linear space, i.e., a vector space X over the real or complex numbers, along with a metric $\rho\colon X\times X\to\mathbb{R}$ such that the addition is continuous with respect the metric ρ, the scalar multiplication is continuous with respect the standard metric of $\mathbb{R}$ or $\mathbb{C}$ and the metric ρ, the metric is translation invariant, i.e., $\rho(x+a,y+a)=\rho(x,y)$ and (X,ρ) is complete. We now apply the previous results to obtain the

Gowers dichotomy for F-spaces. Before doing this we make some remarks on basic sequences in F-spaces. There is an F-space (indeed a quasi-Banach space) which contains no basic sequence [10]. It turns out that there is a dichotomy result for the existence of basic sequences with a very similar flavor to that of the Gowers dichotomy, which has been known for some time.

We will need some background (see [11]). Let X be an F-space and let ρ be an F-norm inducing the topology. A basic sequence $(x_n)_{n=1}^{\infty}$ is called *regular* if $\inf_n \rho(x_n) > 0$. We denote by ω the space of all sequences (i.e., the countable product of lines). The canonical basis of ω is not regular, and ω contains no regular basic sequence. The following result is elementary.

Proposition 5.1. *Suppose X contains no subspace isomorphic to ω. Then given a basic sequence $(x_n)_{n=1}^{\infty}$ we may choose $a_n > 0$ so that $(a_n x_n)_{n=1}^{\infty}$ is regular.*

Proof. Indeed if not we have $\inf_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} \rho(te_n) = 0$. Then some subsequence of $(e_n)_{n=1}^{\infty}$ is equivalent to the canonical basis of ω. ∎

Two subspaces Y, Z of an F-space X are called *separated* if $Y \cap Z = \{0\}$ and the canonical projection $Y + Z \to Y$ is continuous. An F-space is called HI if no two infinite dimensional subspaces are separated.

Proposition 5.2. *Suppose X has a regular basis $(e_n)_{n=1}^{\infty}$. If there exist two separated infinite-dimensional subspaces Y, Z of X then there exist two separated block subspaces of X.*

Proof. Since X is regular the seminorm $\|x\|_\infty = \sup |e_n^*(x)|$ defines a continuous norm on X. Now, fixing $0 < \epsilon < 1/8$ we may inductively define a sequence $(x_n)_{n=1}^{\infty} \in X$ and a block basic sequence $(u_n)_{n=1}^{\infty}$ such that:

(i) $\|x_n\|_\infty = 1$ for all n;
(ii) $\rho(x_n - u_n) + \|x_n - u_n\|_\infty < \epsilon/2^n$ for all n; and
(iii) $x_n \in Y$ for n odd, $x_n \in Z$ for n even.

Note that $\|u_n\|_\infty \geq 1 - \epsilon > 1/2$.

Let P_Y be the canonical projections of $Y + Z$ onto Y with kernel Z. For $v = \sum_{j=1}^n a_j u_j$ in the linear span of the sequence $(u_n)_{n=1}^{\infty}$ we define $Kv = \sum_{j=1}^n a_j (x_j - u_j)$. Then

$$\rho(Kv) \leq \epsilon \max_{1\leq j\leq n} |a_j| \leq 2\epsilon \|v\|_\infty$$

so that K is continuous. Furthermore

$$\|Kv\|_\infty \leq 2\epsilon \max_{1\leq j\leq n} |a_j| \sup \|u_n\|_\infty \leq 4\epsilon \|v\|_\infty.$$

Thus $T = I + K$ extends to a continuous operator $T\colon [u_n]_{n=1}^\infty \to Y + Z$. Now

$$\|Tv\|_\infty \geq (1-4\epsilon)\|v\|_\infty \geq 1/2\|v\|_\infty, \qquad v \in [u_n]_{n=1}^\infty.$$

If $(v_n)_{n=1}^\infty$ is a sequence such that $\lim_{n\to\infty} \rho(Tv_n) = 0$ then $\lim_{n\to\infty} \|Tv_n\|_\infty = 0$ and so $\lim_{n\to\infty} \|v_n\|_\infty = 0$. Thus $\lim_{n\to\infty} \rho(Kv_n) = 0$ and hence $\lim_{n\to\infty} \rho(v_n) = 0$. Hence T is an isomorphism of $[u_n]_{n=1}^\infty$ into $Y+Z$. Consider the operator $S = T^{-1}P_Y T$: then S is a projection of $[u_n]_{n=1}^\infty$ onto $[u_{2n-1}]_{n=1}^\infty$ and the block subspaces $V = [u_{2n-1}]_{n=1}^\infty$ and $W = [u_{2n}]_{n=1}^\infty$ are separated. ∎

Theorem 5.3. *Let X be an F-space with a regular basis containing no unconditional basic sequence. Then X has an HI subspace Y.*

Proof. We assume that X has a regular basis $(e_n)_{n=1}^\infty$.

We now consider the countable dimensional E with Hamel basis $(e_n)_{n=1}^\infty$. Note that the norm $\|\cdot\|_\infty$ on E is continuous with respect to the F-space topology since $(e_n)_{n=1}^\infty$ is regular. For any block basic sequence $(u_n)_{n=1}^\infty$ we say that $(u_n)_{n=1}^\infty$ is somewhat unconditional if the map

$$\sum_{j=1}^\infty a_j u_j \to \sum_{j=1}^\infty (-1)^j a_j u_j$$

(defined for $(a_j)_{j=1}^\infty \in c_{00}$) is continuous for the F-space topology restricted to E. Let σ_0 be the collection of all somewhat unconditional sequences. We claim that with respect to $\mathcal{T}_p$ this set is a Borel subset of $\Sigma_\infty(E)$. Indeed let $(U_m)_{m=1}^\infty$ be a base of open neighborhoods of zero. Let $\sigma_0(m,n)$ be the set of $(u_j)_{j=1}^\infty$ so that

$$\sum_{j=1}^\infty a_j u_j \in U_m \implies \sum_{j=1}^\infty (-1)^j a_j u_j \in \overline{U}_n.$$

Then $\sigma_0(m,n)$ is $\mathcal{T}_p$-closed and $\sigma_0 = \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty \sigma_0(m,n)$.

We define σ to be the subset of $\Sigma_\infty E$ of all block basic sequences $(u_n)_{n=1}^\infty$ such that $\|u_n\|_\infty = 1$ for all n and $(u_n)_{n=1}^\infty$ fails to be somewhat unconditional. Then σ is also Borel in $\mathcal{T}_p$. Furthermore, since X contains no unconditional basic sequence we conclude that σ is large.

Fix some $0 < \epsilon < 1$ and let σ' be the subset of $\Sigma_\infty(E)$ of all sequences $(v_j)_{j=1}^\infty$ such that

$$\inf\left\{ \sum_{j=1}^\infty \big(\|u_j - v_j\|_\infty + \rho(u_j - v_j)\big) \colon (u_j)_{j=1}^\infty \in \sigma \right\} < \epsilon.$$

Note that if $(v_j)_{j=1}^{\infty} \in \sigma'$ there exists $(u_j)_{j=1}^{\infty} \in \sigma$ which is equivalent to $(v_j)_{j=1}^{\infty}$. Hence each $(v_j)_{j=1}^{\infty} \in \sigma'$ fails to be somewhat unconditional.

According to Theorem 4.4 we can find a block subspace V so that σ' is strategically large for V. Let Y be the closure of V. We show that Y is HI. Y has a regular basis $(u_n)_{n=1}^{\infty}$ which is a block basis of $(e_n)_{n=1}^{\infty}$. According to Proposition 5.2 we need only check that if W_1, W_2 are two block subspaces of V then W_1 and W_2 cannot be separated.

Let Φ be the strategy guaranteed by the fact that σ' is strategically large. Then $\Phi(W_1, W_2, W_1, W_2, \dots) = (v_1, v_2, \dots)$ and the sequence $(v_j)_{j=1}^{\infty}$ fails to be somewhat unconditional so that W_1, W_2 are not separated. ∎

Let us now recall the criterion of the existence of basic sequences given in [8] (see also [12]). An F-space X is called *minimal* if there is no strictly weaker Hausdorff topology on X.

Proposition 5.4. *If X is a non-minimal F-space then X contains a regular basic sequence.*

Let us call an infinite-dimensional F-space X *strongly HI (SHI)* if it contains a non-zero vector e so that $e \in L$ for *every* infinite-dimensional closed subspace L of X. We remark that it is possible to consider spaces X which satisfy the slightly stronger condition that any two infinite-dimensional closed subspaces have non-trivial intersection; this condition implies X contains no basic sequence, but it is not clear if it implies that X is SHI. The problem is that we do not know if, under this condition, the intersection of any three infinite-dimensional closed subspaces is non-trivial. This is related to the fact, discussed later, that the sum of two strictly singular operators need not be strictly singular (see the discussion after Theorem 6.1).

Let X be an F-space. We say that a collection $\mathcal{L}$ of closed subspaces of X is a *subspace-filter* in X if each $L \in \mathcal{L}$ is infinite-dimensional and $L_1 \cap L_2 \in \mathcal{L}$ whenever $L_1, L_2 \in \mathcal{L}$; we say that a subspace-filter $\mathcal{L}$ is a *subspace-ultrafilter* if it is not contained properly in any other subspace-filter.

Theorem 5.5. *Let X be an F-space containing no basic sequence. Then X has an SHI-subspace Y.*

Proof. We may assume that X is separable. We pick $\mathcal{L}$ to be a subspace-filter such that $H = \bigcap\{L\colon L \in \mathcal{L}\}$ has minimal dimension $(1 \le \dim H \le \infty)$.

We will argue that $\dim H > 0$. Indeed if $H = \{0\}$ then we define a topology τ on X by taking as a base of neighborhoods sets of the form $U + L$ where U is a neighborhood of zero in the F-space topology and $L \in \mathcal{L}$. If $H = \{0\}$ then τ is Hausdorff. By Proposition 5.4 we have that τ coincides with the original topology. Then we may find a strictly decreasing sequence $L_n \in \mathcal{L}$ so that $L_n \subset \{x\colon \rho(x) < 2^{-n}\}$. If we pick $x_n \in L_n \setminus L_{n+1}$, it is easy to verify that $(x_n)_{n=1}^{\infty}$ is a basic sequence equivalent to the canonical basis of ω.

If $\dim H = \infty$ then it follows from maximality that H has no proper closed infinite-dimensional subspace and so we may take $Y = H$ and y any non-zero element of Y. If $\dim H < \infty$ we first argue by Lindelof's theorem that since X is separable we can find a descending sequence of subspaces $L_n \in \mathcal{L}$ so that $\bigcap L_n = H$. We may suppose this sequence is strictly descending and take $x_n \in L_n \setminus L_{n+1}$ for $n \geq 1$. Let $V_n = [x_k]_{k \geq n}$ so that $V_n \subset L_n$. Suppose W is any closed infinite-dimensional subspace of V_1; then $\dim V_n \cap W = \infty$ for each n. Let $\mathcal{L}'$ be any subspace-ultrafilter containing each V_n and W. Then $\bigcap\{L\colon L \in \mathcal{L}'\} \subset H$ but the inclusion cannot be strict because the original minimality assumption on $\dim H$. Hence $H \subset W$. Thus we can take $Y = V_1$ and $y \in H \setminus \{0\}$. ∎

An examination of the proof shows that we have actually proved a slightly stronger result:

Corollary 5.6. *Let X be an F-space containing no basic sequence. Then X has an SHI-subspace Y with the property that if E is the intersection of all infinite-dimensional subspaces of Y then there is a descending sequence of infinite-dimensional subspaces $(L_n)_{n=1}^{\infty}$ of Y with $\bigcap_{n=1}^{\infty} L_n = E$.*

We are now ready to establish the full force of the Gowers dichotomy for F-spaces.

Theorem 5.7. *Let X be an F-space. If X contains no unconditional basic sequence, then X contains an HI subspace.*

Proof. If X contains no basic sequence then X contains a SHI subspace (Theorem 5.5). So we may assume X has a basis. Clearly X cannot contain a copy of ω so we can assume the basis is regular (Proposition 5.1). Now apply Theorem 5.3. ∎

We conclude this section with:

Theorem 5.8. *Let X be an HI F-space. Suppose X has a closed infinite-dimensional subspace containing no basic sequence. Then X contains no basic sequence.*

Proof. We will show that if $(V_n)_{n=1}^{\infty}$ is any descending sequence of closed infinite-dimensional subspaces of X then $\bigcap_{n=1}^{\infty} V_n \neq \{0\}$. We use Corollary 5.6 to deduce the existence of a descending sequence of infinite-dimensional closed subspaces $(L_n)_{n=1}^{\infty}$ such that, if $E = \bigcap_{n=1}^{\infty} L_n$, then

$E \neq \{0\}$ and E is contained in any infinite-dimensional subspace of L_1. Consider the sequence $(L_n \cap V_n)_{n=1}^{\infty}$. Then if $\dim L_n \cap V_n = \infty$ for all n we have $E \subset \bigcap_{n=1}^{\infty} L_n \cap V_n \subset \bigcap_{n=1}^{\infty} V_n$.

If not then there exists n_0 such that $\dim(L_n \cap V_n)$ is finite and constant for $n \geq n_0$. Hence $L_n \cap V_n = F$ some fixed finite dimensional subspace for $n \geq n_0$. We show $\dim F > 0$. If for some $n \geq n_0$ we have $L_n \cap V_n = \{0\}$ then $L_n + V_n$ cannot be closed since X is HI. Thus there are sequences $(x_k)_{k=1}^{\infty}$ in L_n and $(v_k)_{k=1}^{\infty} \in V_n$ so that $\lim \rho(x_k + v_k) = 0$ but $\rho(x_k) \geq \delta > 0$ for all k. Consider the metric topology on L_n defined by the F-norm $x \to d(x, V_n) := \inf\{\rho(x+v) \colon v \in V\}$. This topology is Hausdorff on L_n and strictly weaker than the ρ-topology. Hence L_n contains a basic sequence by Proposition 5.4, and this is a contradiction. Hence $\dim F > 0$ and $F \subset \bigcap_{n=1}^{\infty} V_n$. ∎

6. Strictly singular maps

In [7] the following Theorem is shown:

Theorem 6.1. *Let X be a complex Banach space. If X is HI then every bounded linear operator $T \colon X \to X$ is of the form $T = \lambda I + S$ where S is strictly singular.*

We do not know whether such a theorem can hold for a complex F-space but we show that it holds equally for complex quasi-Banach spaces. There are some small wrinkles in the proof as the reader will see.

From now on we will deal with quasi-Banach space X (or Y, etc.) with a given quasi-norm which is assumed to be p-subadditive (for a suitable $0 < p \leq 1$), i.e.,

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \qquad x, y \in X.$$

A linear operator $T \colon X \to Y$ is an *isomorphic embedding* if there exists $c > 0$ so that $\|Tx\| \geq c\|x\|$ for $x \in X$. T is called *strictly singular* if $T|_V$ fails to be an isomorphic embedding for every infinite-dimensional subspace V of X. T is called *semi-Fredholm* if $\ker T$ is finite-dimensional and $T(X)$ is closed. T is called *Fredholm* if T is semi-Fredholm and $\dim Y/T(X) < \infty$.

T is semi-Fredholm if and only if for every bounded sequence $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \|Tx_n\| = 0$ we can extract a convergent subsequence. Thus it is clear the restriction of a semi-Fredholm operator to an infinite-dimensional closed subspace remains semi-Fredholm.

Let us make some remarks. Suppose X is a SHI space and let E_X be the intersection of all closed infinite-dimensional subspaces of X. If $\dim E_X = \infty$ then E_X is an *atomic space*, i.e., it has no proper closed infinite-dimensional subspace. The existence of atomic spaces is still open (the only known results in this direction are in [15]). However it is known that there exist quasi-Banach spaces X for which E_X is finite-dimensional and non-trivial, even with $\dim E_X > 1$ ([9, Theorem 5.5]). The quotient map $Q\colon X \to X/E_X$ is then both semi-Fredholm and strictly singular (this cannot happen for operators on Banach spaces). Furthermore if $\dim E_X > 1$ then let L_1, L_2 be two distinct one-dimensional subspaces of E_X. Then the quotient maps $Q_1\colon X \to X/L_1$ and $Q_2\colon X \to X/L_2$ are both strictly singular and semi-Fredholm. However the map $x \to (Q_1 x, Q_2 x)$ from X into $X/L_1 \oplus X/L_2$ is an isomorphism. Thus the sum of two strictly singular operators need not be strictly singular!

The key fact we will need is the following:

Theorem 6.2. *Let X be an infinite-dimensional complex quasi-Banach space and suppose $T\colon X \to X$ is a bounded operator. Then there exists $\lambda \in \mathbb{C}$ so that $T - \lambda I$ is not semi-Fredholm.*

This Theorem is proved for Banach spaces by Gowers and Maurey [7]. The proof for quasi-Banach spaces requires some additional tricks. These tricks are necessitated by the fact that finite-dimensional subspaces are not always complemented.

We list the relevant facts we need:

Proposition 6.3. *If X is a complex quasi-Banach space and $T\colon X \to X$ is a bounded linear operator then $Sp(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I$ is not invertible$\}$ is a non-empty compact set and $\max_{\lambda \in Sp(T)} |\lambda| = \lim_{n\to\infty} \|T^n\|^{1/n}$.*

This is due to Żelazko [16]. We point out that the key ideas in the proof involve a reduction to the Banach algebra case. One starts with the fact ([16]) that on a commutative quasi-Banach algebra the formula $r(x) = \lim_{n\to\infty} \|T^n\|^{1/n}$ defines a seminorm. Using this one can prove the Gelfand–Mazur theorem (see e.g. [11]) in this context and develop the basic theory of commutative quasi-Banach algebras. The Proposition is obtained by looking at the double commutant of T.

Proposition 6.4. *Let X be a complex quasi-Banach space and let $\mathcal{G}_1$ denote the subset of $\mathcal{L}(X)$ consisting of all isomorphic embeddings and $\mathcal{G}_2$ be the collection of all surjections. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are both open sets and $\mathcal{G}_1 \cap \mathcal{G}_2$ is a clopen subset relative to $\mathcal{G}_1$ and relative to $\mathcal{G}_2$.*

See [11, pp. 132–134].

Proposition 6.5. *Let X be an infinite-dimensional complex Banach space and suppose $T\colon X \to X$ is quasi-nilpotent, i.e., $Sp(T) = \{0\}$. Then T cannot be semi-Fredholm.*

See [7, Lemma 19]. We will now need to prove this Proposition for a general complex quasi-Banach space. We do this in several very simple steps. Assume throughout that X is an infinite-dimensional complex quasi-Banach space.

Lemma 6.6. *Suppose $T\colon X\to X$ is any bounded operator and $\lambda\in\partial Sp(T)$. Then $T-\lambda I$ can be neither an isomorphic embedding nor a surjection.*

Proof. This follows from Proposition 6.4. ∎

Lemma 6.7. *Suppose X has trivial dual. If $T\colon X\to X$ is quasi-nilpotent then T cannot be Fredholm.*

Proof. If $T(X)$ has finite codimension in X then T is onto in this case. We then use Lemma 6.6. ∎

Lemma 6.8. *If X is any infinite-dimensional complex quasi-Banach space and $T\colon X\to X$ is quasi-nilpotent then T cannot be Fredholm.*

Proof. Denote by X^* the dual of X; this is a Banach space but it can be quite small (even $\{0\}$). We assume $X^*\neq\{0\}$ as this case is covered in Lemma 6.7. Assume $T\colon X\to X$ is quasi-nilpotent and Fredholm. Then $T^*\colon X^*\to X^*$ is Fredholm. In fact $T^*(X^*)=\ker(T)^\perp$; this depends on the fact that every continuous linear functional y^* on $T(X)$ can be extended to $x^*\in X^*$ since $\dim X/T(X)<\infty$. Since $\|(T^*)^n\|\le\|T^n\|$ the spectral radius formula shows that T^* is quasi-nilpotent. By Proposition 6.5 we must have $\dim X^*<\infty$. Let $X_0=\{x\in X : x^*(x)=0\ \forall\ x^*\in X^*\}$. Then X_0 is invariant for T and of finite-codimension in X. Clearly $X_0^*=\{0\}$ and $T|_{X_0\to X_0}$ remains Fredholm so we can apply Lemma 6.7 to get a contradiction. ∎

Lemma 6.9. *If X is any infinite-dimensional complex quasi-Banach space and $T\colon X\to X$ is quasi-nilpotent then T cannot be semi-Fredholm.*

Proof. Assume T is semi-Fredholm. Then by a Baire Category argument there exists $x\in X$ so that $T^nx\neq 0$ for every $n\in\mathbb{N}$. Let $Y=[T^nx]_{n=1}^\infty$. Then $T\colon Y\to Y$ is Fredholm and remains quasi-nilpotent (using Proposition 6.3). Clearly $T|_{Y\to Y}$ is not nilpotent so $\dim Y=\infty$. This is a contradiction by Lemma 6.8. ∎

Proof of Theorem 6.2. The remaining steps in the proof of Theorem 6.2 are very similar to those in [7] for the Banach space case. Assume $T-\lambda I$ is semi-Fredholm for all $\lambda\in\mathbb{C}$. We suppose $\lambda\in\partial\mathrm{Sp}(T)$ is an accumulation point of $\partial\mathrm{Sp}(T)$. Let $\lambda_n\to\lambda$ with $\lambda_n\neq\lambda$ and $\lambda_n\in\partial\mathrm{Sp}(T)$. Each λ_n is an eigenvalue of T (by Lemma 6.6, since $T-\lambda_n I$ is semi-Fredholm), say with eigenvector x_n. Let $Y=[x_n]_{n=1}^\infty$. Then Y is invariant for T and $\lambda\in\partial\,\mathrm{Sp}\,(T|_{Y\to Y})$. However $(T-\lambda I)|_{Y\to Y}$ has dense range and is semi-Fredholm. Hence $(T-\lambda I)|_{Y\to Y}$ is surjective and we have a contradiction by Lemma 6.6.

It follows that $\partial\,\mathrm{Sp}\,(T)$ has no accumulation points and hence is a finite set. Thus $\mathrm{Sp}\,(T)$ is also finite say $\mathrm{Sp}\,(T)=\{\lambda_1,\dots,\lambda_n\}$. Then $S=\prod_{k=1}^n(T-\lambda_k I)$ is semi-Fredholm and $\mathrm{Sp}\,(S)=\{0\}$. This contradicts Lemma 6.9. ∎

Theorem 6.10. *Let X be an infinite-dimensional complex quasi-Banach space. If $T\colon X\to X$ is strictly singular then T cannot be semi-Fredholm.*

Remark. Note that this is false for operators $T\colon X\to Y$ by the remarks above.

Proof. In fact $T-\lambda I$ is always semi-Fredholm if $\lambda\neq 0$ (Theorem 7.10 of [11]). The result follows from Theorem 6.2. ∎

Theorem 6.11. *Let X be an infinite-dimensional complex quasi-Banach space. If $T\colon X\to X$ is a bounded linear operator then exactly one of the following two conditions holds:*

(i) For every $\epsilon>0$ there is an infinite-dimensional closed subspace V of X such that $\|T|_V\|<\epsilon$.
(ii) T is semi-Fredholm.

If X is HI then (i) is equivalent to:

(i′) T is strictly singular.

Proof. Assume (ii). Then there is a constant $c>0$ so that $\|Tx\|\geq cd(x,F)$ for $x\in X$, where $F=\ker T$. If V is an infinite-dimensional closed subspace we can find a sequence $(v_n)_{n=1}^\infty$ in the unit ball so $\|v_m-v_n\|\geq 1/2$ for $m\neq n$. Assuming that the norm is p-convex, by a simple compactness argument we can then show the existence of a pair $m\neq n$ so that $(d(v_m,F)^p+d(v_n,F)^p)^{1/p}\geq 1/4$. Hence $\|Tv_m\|^p+\|Tv_n\|^p\geq(1/4)^p c^p$. This implies a lower bound on $\|T|_V\|$.

Now assume (ii) fails and that $F=\ker(T)$ is finite dimensional. Then T factors in the form $T=T_0Q$ where $Q\colon X\to X/F$ is the quotient map and $T_0\colon X/F\to X$ is one-one but not an isomorphic embedding. Then there is a normalized sequence $\xi_n\in X/F$ so that $\|T_0\xi_n\|<2^{-n}$. Now using Theorem 4.6 of [11] we can assume by passing to a subsequence that $(\xi_n)_{n=1}^\infty$ satisfies an estimate

$$\max_{1\leq k\leq n}|a_k|\leq C\left\|\sum_{k=1}^n a_k\xi_k\right\|,\qquad a_1,\dots,a_n\in\mathbb{C}.$$

In particular if $V_k = Q^{-1}[\xi_j]_{j\geq k}$ then each V_k is infinite-dimensional and $\|T|_{V_k}\| \to 0$. Thus (i) holds.

Now assume X is HI. Suppose T satisfies (i) and is not strictly singular. Then there is an infinite-dimensional subspace W so that $\|Tw\| \geq \delta\|w\|$ for $w \in W$ where $\delta > 0$. Pick $\epsilon = \delta/2$ and then choose V as in (i) for this ϵ. Clearly $V \cap W = \{0\}$.

Now assume $v \in V$, $w \in W$ with $\|v+w\| = 1$. Then

$$\begin{aligned}
\|v\|^p &\leq 1 + \|w\|^p \\
&\leq 1 + 2^{-p}\|v\|^p + \|w\|^p - 2^{-p}\|v\|^p \\
&\leq 1 + 2^{-p}\|v\|^p + \delta^{-p}\|Tw\|^p - 2^{-p}\epsilon^{-p}\|Tv\|^p \\
&= 1 + 2^{-p}\|v\|^p + \delta^{-p}(\|Tw\|^p - \|Tv\|^p) \\
&\leq 1 + 2^{-p}\|v\|^p + \delta^{-p}\|T(v+w)\|^p \\
&\leq 1 + 2^{-p}\|v\|^p + \delta^{-p}\|T\|^p.
\end{aligned}$$

Thus

$$\|v\| \leq \left(\frac{1 + \delta^{-p}\|T\|^p}{1 - 2^{-p}}\right)^{1/p}.$$

This contradicts the fact that X is HI.

Conversely if T is strictly singular it cannot be semi-Fredholm by Theorem 6.10 and so (i) must hold. ∎

Theorem 6.12. *Let X be an infinite-dimensional complex quasi-Banach space. If X is HI then every bounded linear operator $T: X \to X$ is of the form $T = \lambda I + S$ where S is strictly singular.*

Proof. There exists λ so that $T - \lambda I$ is not semi-Fredholm by Theorem 6.2. By Theorem 6.11 this means $T - \lambda I$ is strictly singular. ∎

In the case when X is SHI this result is much simpler. Indeed we have:

Theorem 6.13. *Let X be an SHI space and suppose E is the intersection of all infinite-dimensional subspaces of X. Let $Q: X \to X/E$ be the quotient map (which is strictly singular). Then if $T: X \to X$ is a bounded operator, there exists $\lambda \in \mathbb{C}$ and a bounded operator $S: X/E \to X$ so that $T = \lambda I + SQ$.*

Proof. Let us first give a simpler proof of Theorem 6.12. It is clearly that if $R: X \to X$ is an invertible operator then $R(E) \subset E$ and this implies that E is invariant for all operators on X. If E is atomic then E is rigid ([11, Theorem 7.22, p. 155]). Otherwise E is finite-dimensional. In either case $T|_E$ has an eigenvalue λ and so $T-\lambda I$ factors through a quotient map $Q'\colon X \to X/F'$ where F' is a non-trivial subspace of E. Hence $T-\lambda I$ is strictly singular.

Now using Theorem 6.11 it is clear any strictly singular operator on X vanishes on E and so we get the desired factorization. ∎

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