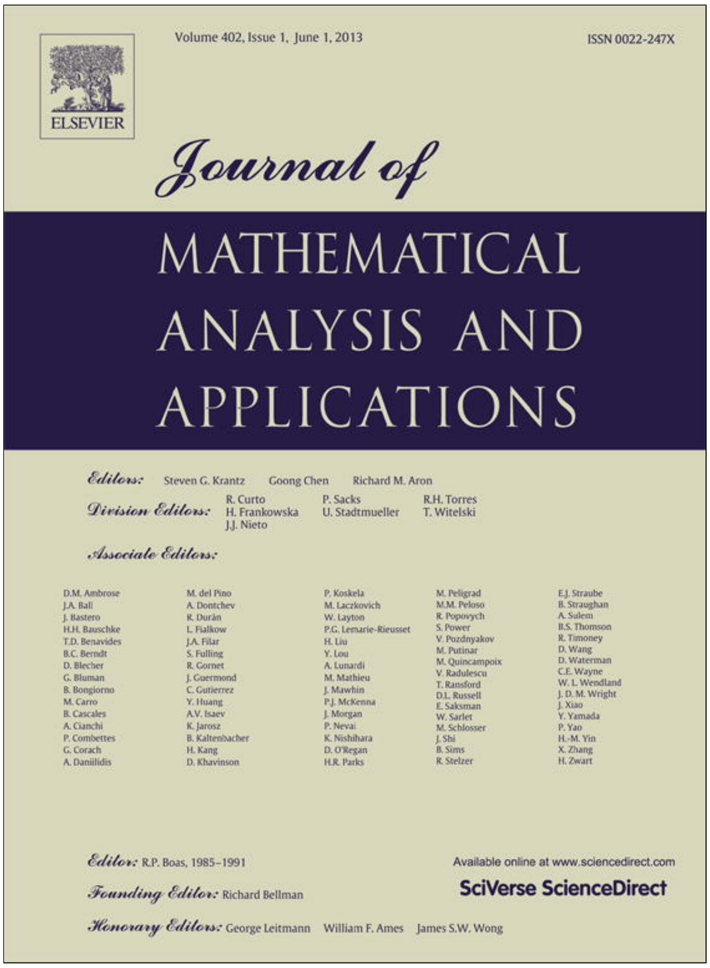
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Compactly uniformly convex spaces and property (β) of Rolewicz

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ABSTRACT

We study different facets of property (β) of Rolewicz. We remark that the notions of compact uniform convexity and property (β) are isometrically equivalent and present new examples of spaces with that property. An observation is made that the property (β) can be formulated in terms of graphs and an estimate of the (β)-modulus is also given.

1. Introduction

In the 1980's Stefan Rolewicz introduced a geometric property (β) [22,23] which is intermediate between uniform convexity and nearly uniform convexity. It turned out that it defined an isomorphically different class of spaces [10,17,11,12].

Recently, Lima and Randrianarivony [16] pointed out the role of the property (β) in nonlinear quotient problems, and answered a ten-year-old question of Bates, Johnson, Lindenstrauss, Preiss and Schechtman [2]. Lima and Randrianarivony used an isometric characterization of (β) [13] and estimates of the (β)-modulus for ℓ_p-spaces [1]. The results from [16] are further generalized in [8].

Independently, the last two authors [21] introduced recently the notion of compact uniform convexity in connection to the study of metric projections. They proved that in the class of compactly uniformly convex spaces, the set of points x for which the best approximation problem to a nonempty closed set A is generalized well-posed and has a complement which is a σ-porous set. Their result is an isometric generalization of a theorem of De Blasi, Myjak and Papini [5], proved in the setting of uniformly convex spaces.

In the present paper we show that property (β) of Rolewicz and compact uniform convexity are isometrically equivalent. Thus, by [10,17,11,12], the above mentioned result from [21] is isomorphically stronger than the corresponding result from [5].

Reformulation of a characterization of property (β) from [13] is given with the help of graphs in Section 3.

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The typical examples of spaces with property (β) [10,17] were ℓ_p-sums of finite dimensional spaces for $1 < p < \infty$. In Section 4 we give some new examples of spaces with this property, namely the injective and projective tensor products of ℓ_p and ℓ_q, for p and q in a determined range. We obtain property (β) for these spaces somewhat indirectly, by appealing to the result from [12] that a space which is simultaneously nearly uniformly smooth and nearly uniformly convex has property (β). To apply this result, we first compute exactly the power type of the moduli of nearly uniform smoothness and nearly uniform convexity for these spaces.

Finally, in Section 5, we study the modulus $\overline{\beta}(t)$. It is known [12] that if the norm of a Banach space X is both nearly uniformly convex (NUC) and nearly uniformly smooth (NUS), then the norm has property (β). We prove an estimate for the (β)-modulus $\overline{\beta}(t)$, provided that the estimates for the NUC and NUS moduli are of certain power types.

2. Preliminaries and equivalence

Let $(X, \|\cdot\|)$ be a real Banach space with topological dual X^*. As usual B_X and S_X will stand for the closed unit ball and unit sphere in X respectively, and more generally, $B[x, r]$ and $S(x, r)$ will be used for the closed ball centered at $x \in X$ and radius $r > 0$ and the corresponding sphere of this ball. As usual $B(x, r)$ is reserved for the open ball centered at x and with radius $r > 0$. The origin in X is denoted by θ. For given $x, y \in X$, the closed line segment between x and y is designated by $[x, y]$, and (x, y) is the set $[x, y] \setminus \{x, y\}$.

Let us recall that the norm $\|\cdot\|$ (or, equivalently, the space X) is called *locally uniformly convex* (briefly, LUC) if, whenever, $x \in S_X$ and $(x_n)_n \subset S_X$ are such that $\lim_n \|x + x_n\| = 2$, then the sequences $(x_n)_n$ converges to x. Based on this definition, the following generalization was employed in [27,18] for the study of certain properties of metric projections: a Banach space X, $(\dim X \geq 2)$, is called *compactly locally uniformly convex* (in brief, CLUC) if, whenever $x \in S_X$ and $(x_n)_n \subset S_X$ are such that $\lim_n \|x + x_n\| = 2$, then $(x_n)_n$ has a convergent subsequence.

Evidently, every locally uniformly convex space is compactly locally uniformly convex. Reciprocally, it can be seen that, if the space X is CLUC and also *strictly convex* (the latter as usual means that S_X does not contain line segments) then X is LUC.

In order to present a geometric characterization of the above generalized property, let us recall that the *Kuratowski index of non-compactness* $\alpha(A)$ for a set $A \subset X$ is the infimum of all $\varepsilon > 0$ such that A can be covered by a finite number of sets with diameters less than ε. It is known that $\alpha(A) = 0$ if and only if A is relatively compact. The generalized Cantor lemma says that, if $(A_n)_n$ is a nested sequence of nonempty closed sets in a Banach space X, such that $\alpha(A_n) \to 0$, then $\bigcap_n A_n$ is a nonempty compact set of X.

Let now $x \in S_X$, and $\delta \in [0, 1]$. Consider the following "cap" generated by x and δ:

$$\mathrm{Cap}[x, \delta] = \left\{ y \in S_X : \left\| \frac{x + y}{2} \right\| \geq 1 - \delta \right\}.$$

This is the (nonempty) set of points y on the sphere S_X such that the mid-points of the segments $[x, y]$ do not lie deeper inside B_X than $1 - \delta$. The set $\mathrm{Cap}[x, \delta]$ is obviously a closed subset of S_X. These sets are monotone with respect to δ, that is

$$\mathrm{Cap}[x, \delta_1] \subset \mathrm{Cap}[x, \delta_2] \quad \text{whenever } 0 \leq \delta_1 \leq \delta_2.$$

Of course, one can define similar "caps" on any sphere $S(x, r)$ of any ball in the space and the properties which we will mention below are true for such caps as well. The following fact was observed in [21].

Lemma 2.1 (*[21, Lemma 2.1 and Remark 2.2]*). *The Banach space X is CLUC if and only if for any $x \in S_X$ one has $\lim_{\delta \downarrow 0} \alpha(\mathrm{Cap}[x, \delta]) = 0$.*

Let us mention that evidently, in a CLUC Banach space X, the set $\mathrm{Cap}[x, 0] = \bigcap_{0<\delta\leq 1} \mathrm{Cap}[x, \delta]$ is nonempty and compact for every $x \in S_X$.

The above characterization of CLUC Banach spaces suggested the following definition in [21].

Definition 2.2. The Banach space X is called *compactly uniformly convex* (shortly, CUC), if we have $\lim_{\delta \downarrow 0} \alpha(\mathrm{Cap}[x, \delta]) = 0$ uniformly on $x \in S_X$.

It is evident that any compactly uniformly convex Banach space is also a CLUC space. It is clear also that any finite dimensional normed space is compactly uniformly convex. It can be seen that any uniformly convex space is compactly uniformly convex as well: we recall that $(X, \|\cdot\|)$ is *uniformly convex* if for any $\varepsilon \in (0, 2]$ there is some $\delta \in (0, 1)$ so that $x, y \in S_X$ with $\|x - y\| > \varepsilon$ implies $\|x + y\| < 2(1 - \delta)$.

The compact uniform convexity property was introduced in [21] as a strengthening of the CLUC property, again with the aim to study properties of metric projections, and more specifically, to investigate the descriptive nature of the set of points of existence and stability of best approximations to closed sets in Banach spaces. The name of the property is derived from the notion of compactly locally uniformly convex space.

Although the main goal of the paper [21] was to study metric projections, several properties of the above notion were obtained and some examples were given. In particular, it was shown that any CUC space X is a nearly uniformly convex space [21, Proposition 2.6]. We recall that a Banach space X is called *nearly uniformly convex* (NUC, in short) [9], if for any $\varepsilon > 0$ there is $\delta \in (0, 1)$ such that for each sequence $(x_n)_n \subset B_X$ with $\mathrm{sep}(x_n) \geq \varepsilon$ it follows $\mathrm{co}(x_n) \cap (1-\delta)B_X \neq \emptyset$. Here, $\mathrm{sep}(x_n)$ is defined as

$$\mathrm{sep}(x_n) := \inf\{\|x_m - x_l\| : m \neq l\},$$

and $\mathrm{co}(x_n)$ is the convex hull of the elements of the sequence $(x_n)_n$.

In particular, every CUC space is reflexive. Example 2.9 from [21] shows that there are nearly uniformly convex norms which are not CLUC norms (and thus are not compactly uniformly convex either).

We will see below that the compact uniform property is equivalent to another geometric property proposed and studied by Rolewicz in [22,23] in the 1980's, called property (β). To present it, let us recall that given $x \in X \setminus B_X$, the *drop generated by* x is the set $D(x, B_X) := \mathrm{co}(\{x\} \cup B_X)$. Denote by $R(x, B_X)$ the part of the drop which is not in the unit ball, that is $R(x, B_X) := D(x, B_X) \setminus B_X$. Rolewicz proved in [22] that the space X is uniformly convex if and only if for any $\varepsilon > 0$ there is $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies that $\mathrm{diam}(R(x, B_X)) < \varepsilon$. And then, related to this, he introduced in [23] the following property: a Banach space is said to have *property* (β) if for any $\varepsilon > 0$ there is $\delta > 0$ so that $1 < \|x\| < 1 + \delta$ implies that $\alpha(R(x, B_X)) < \varepsilon$.

It was shown in [23] that any uniformly convex space has property (β) and that spaces with property (β) are nearly uniformly convex spaces. It was proved that the class of Banach spaces in which there is an equivalent norm with property (β) contains strictly the class of superreflexive spaces (cf. [10,17]) and is contained strictly in the class of spaces in which there is an equivalent NUC norm (cf. [11]). The isomorphic characterization of property (β) for Banach spaces with a basis was given later in [12].

In order to prove the equivalence between the compact uniform convexity and property (β) we need two lemmas.

Lemma 2.3. *Let X be normed space with $\dim X \geq 2$. For every $x \in S_X$, and $\delta > 0$ such that $\sqrt{\delta} + 2\delta \leq 2^{-1}$ we have:*

(i) $2\sqrt{\delta}x + (1 + 2\sqrt{\delta})^{-1}\mathrm{Cap}[x, \delta] \subset R((1 + 2\sqrt{\delta})x, B_X)$;
(ii) $\mathrm{diam}\mathrm{Cap}[x, \delta] \leq (1 + 2\sqrt{\delta})\mathrm{diam}R((1 + 2\sqrt{\delta})x, B_X)$;
(iii) $\alpha(\mathrm{Cap}[x, \delta]) \leq (1 + 2\sqrt{\delta})\alpha(R((1 + 2\sqrt{\delta})x, B_X))$.

Proof. It suffices to prove the assertion (i) only, as (ii) and (iii) follow immediately from it. Put for brevity $s = 2\sqrt{\delta}$ and take $z \in sx + (1 + s)^{-1}\mathrm{Cap}[x, \delta]$. There exists $y \in \mathrm{Cap}[x, \delta]$ such that $z = sx + (1 + s)^{-1}y$. Certainly, $\|y\| = 1$ and from the presentation $z = s(1 + s)^{-1}(1 + s)x + (1 + s)^{-1}y$ it is evident that $z \in D((1 + s)x, B_X)$. Also,

$$(1 + s)z = (s + s^2)x + y = x + y - (1 - s - s^2)x,$$
$$2 - \frac{s^2}{2} \leq \|x + y\| \leq (1 + s)\|z\| + 1 - s - s^2,$$

whence $(1 + s)\|z\| \geq 1 + s + s^2/2 > 1 + s$, i.e. $\|z\| > 1$ and $z \in R((1 + 2\sqrt{\delta})x, B_X)$. □

In a normed space X, $\dim X \geq 2$, for $x \in X$ with $\|x\| > 1$ define the set $\tilde{R}(x, B_X) = \mathrm{co}(\{x\} \cup B_X) \setminus B(\theta, 1)$. Certainly, $R(x, B_X) \subset \tilde{R}(x, B_X)$ and generally $\tilde{R}(x, B_X) \neq \overline{R(x, B_X)}$ (both $\mathrm{diam}\tilde{R}(x, B_X) > \mathrm{diam}R(x, B_X)$, and $\alpha(\tilde{R}(x, B_X)) > \alpha(R(x, B_X))$ in infinite dimensions, might occur).

Nevertheless, we have

Lemma 2.4. *Let X be a normed space, $\dim X \geq 2$. For every $x \in S_X$ and $\delta \in (0, 1)$ the following hold:*

(i) $\tilde{R}((1 + \delta)x, B_X) \subset \mathrm{Cap}[x, \delta/2] + \delta B_X$;
(ii) $\mathrm{diam}\tilde{R}((1 + \delta)x, B_X) \leq \mathrm{diam}\mathrm{Cap}[x, \delta/2] + 2\delta$;
(iii) $\alpha(\tilde{R}((1 + \delta)x, B_X)) \leq \alpha(\mathrm{Cap}[x, \delta/2]) + 2\delta$.

Proof. The assertion (i) relies on the following fact from the planar geometry:

Fact: *In a 2-dimensional normed space P, three points x, y, and z are given such that $\|x\| > 1$, $\|y\| = \|z\| = 1$, and both $(x, y) \subset P \setminus B_P$ and $(x, z) \subset P \setminus B_P$ are true. Then for every $u \in S_P \cap \mathrm{co}\{x, y, z\}$ it follows $(x, u) \subset P \setminus B_P$.*

In order to verify this fact, assume the contrary: There are $u \in S_P$ such that $u \in \Delta = \mathrm{co}\{x, y, z\}$ and $v \in (x, u) \cap B_P$. Notice first, that $u \in (y, z)$: Indeed, u is not in the interior of $\mathrm{co}\{y, z, v\}$, as $\|u\| = 1$, and certainly $u \notin [y, v] \cup [z, v]$. Observe next, that v and the origin θ are in the same open half-plane defined by the line l_{yz} through y and z: This is so, since otherwise (θ, v) meets l_{yz} at an interior point w of B_P and then u which is between y and z should be interior to B_P.

Thus θ and v, as well as x, and $-y$, and $-z$, are in one and the same open half-plane defined by l_{yz}. The segment $[-y, -z]$ is not contained in Δ. Assume, without loss of generality, that $-y \notin \Delta$ and apply the planar version of Radon theorem with respect to the set $Q = \{x, y, -y, z\}$. Then Q is partitioned on two sets Q_1 and Q_2 such that $\mathrm{co}Q_1 \cap \mathrm{co}Q_2 \neq \emptyset$. Since no point from Q is in the convex hull of the other three, and since $[x, -y] \cap [y, z] = \emptyset$ then

$$[x, y] \cap [-y, z] \neq \emptyset \quad \text{or} \quad [x, z] \cap [y, -y] \neq \emptyset,$$

which gives a contradiction and establishes the fact.

Let now $z \in \tilde{R}((1+\delta)x, B_X)$. Put $z' = z/\|z\|$. Certainly, $\|z - z'\| < \delta$. Having in mind the aforementioned fact, for $w = 2^{-1}z' + 2^{-1}(1+\delta)x$, conclude that $\|w\| \geq 1$. On the other hand $w = 2^{-1}(x+z') + 2^{-1}\delta x$ whence $1 - \delta/2 \leq 2^{-1}\|x+z'\|$, i.e. $z' \in \mathrm{Cap}[x, \delta/2]$ and (i) holds true.

The assertions (ii) and (iii) follow from (i). □

With Lemmas 2.3 (iii) and 2.4 (iii) in hand, we establish the equivalence between the compact uniform convexity and (β) property.

Proposition 2.5. *The Banach space X is compactly uniformly convex if and only if it satisfies (β) property.*

The localized property (β) for a Banach space X (called L-β) is defined in [25]: for any x with $\|x\| = 1$, $\alpha(R(tx, B_X)) \longrightarrow 0$ as $t \to 1$, $t > 1$. It follows from Lemmas 2.1, 2.3 (iii), and 2.4 (iii) that L-β coincides with CLUC.

Proposition 2.6. *The Banach space X is compactly locally uniformly convex if and only if it satisfies L-β property.*

Remark. In view of Lemmas 2.4 (iii), and 2.3 (iii), both properties (β) and L-β can be defined by using the sets $\tilde{R}(x, B_X)$ in place of $R(x, B_X)$.

In [14] a characterization of property (β) by lens sets is given. Given $x, y \in X$, $r > 0$, $y \in B(x, r)$, $y \neq x$, $\sigma \in (0, 2\|x-y\|)$, recall that the lens set depending on x, r, y, and σ is defined as follows

$$\mathrm{Lens}(x, y, r, \sigma) = B[y, r - \|y - x\| + \sigma] \setminus B(x, r).$$

In case $x = \theta$ and $r = 1$ we write (whenever there is no ambiguity) $\mathrm{Lens}(y, \sigma)$ instead of $\mathrm{Lens}(x, y, r, \sigma)$, i.e.

$$\mathrm{Lens}(y, \sigma) = B[y, 1 - \|y\| + \sigma] \setminus B(\theta, 1), \quad 0 < \|y\| < 1,\ \sigma \in (0, 2\|y\|).$$

The lens sets appear naturally in approximation problem. We mention only a result in [24], namely, the observation that in uniformly convex spaces 'uniformly small lenses have uniformly small diameters', more precisely, for each $\varepsilon > 0$ and each $t \in (0, 1)$ there is $\delta > 0$ such that for every x, $\|x\| = 1$, we have $\mathrm{diam}(\mathrm{Lens}(tx, \delta)) < \varepsilon$. A more detailed reference concerning lenses is found in [14,4].

Theorem 2.7 *([14, Theorem 2]). Let X be a Banach space. If the norm has the property (β), then for each $0 < t < 1$ and each $\varepsilon > 0$ there is a $\delta > 0$ so that for every $x \in S_X$, $\alpha(\mathrm{Lens}(tx, \delta)) < \varepsilon$. Conversely, if for some $0 < t < 1$ we have that $\alpha(\mathrm{Lens}(tx, \delta)) \to 0$ as $\delta \to 0$ uniformly in x, then the norm has the property (β).*

Theorem 2.7 is established by a proposition (Proposition 1, [14]) giving quantitative relations between the sets $\mathrm{Lens}(tx, \delta)$ and $R((1+\delta)x, B_X)$ for $\|x\| = 1$, $0 < t < 1$, and small $\delta > 0$. Similar relations between 'drop remainders' and 'caps' were given by Lemmas 2.3 and 2.4 in the present paper. In [21] (Lemma 3.1 (ii)), it is shown that

$$\alpha(\mathrm{Lens}(tx, \delta)) \leq (1-t)\alpha\left(\mathrm{Cap}\left[x, \frac{\delta}{2t}\right]\right) + 2\delta, \quad \|x\| = 1,\ 0 < t < 2^{-1},\ \sigma \in (0, 2t). \tag{2.1}$$

The missing connection between 'caps' and 'lens sets', the inverse type inequality to (2.1), is contained in the next

Lemma 2.8. *Let X be a normed space, $\dim X \geq 2$, and $\delta > 0$ be such that $2\delta + \sqrt{2\delta} \leq 1$ (i.e. $1 + \sqrt{2\delta}$ is less than or equal to the golden ratio!). Then for each $x \in S_X$*

(i) $\sqrt{2\delta}x + (1+\sqrt{2\delta})^{-1}\mathrm{Cap}[x, \delta] \subset \mathrm{Lens}(\sqrt{2\delta}x, 2\delta(1+\sqrt{2\delta})^{-1})$;
(ii) $\mathrm{diam}(\mathrm{Cap}[x, \delta]) \leq (1+\sqrt{2\delta})\mathrm{diam}(\mathrm{Lens}(\sqrt{2\delta}x, 2\delta(1+\sqrt{2\delta})^{-1}))$;
(iii) $\alpha(\mathrm{Cap}[x, \delta]) \leq (1+\sqrt{2\delta})\alpha(\mathrm{Lens}(\sqrt{2\delta}x, 2\delta(1+\sqrt{2\delta})^{-1}))$.

Proof. Denote for brevity $s = \sqrt{2\delta}$ and take $z \in sx + (1+s)^{-1}\mathrm{Cap}[x, \delta]$. There exists $y \in \mathrm{Cap}[x, \delta]$ such that $z = sx + (1+s)^{-1}y$. Then

$$(1+s)z = (s+s^2)x + y = x + y - (1-s-s^2)x.$$

Since

$$2 - s^2 \leq \|x+y\| \leq (1+s)\|z\| + (1-s-s^2),$$

we have $(1+s)\|z\| \geq 1+s$, and consequently $\|z\| \geq 1$.

On the other hand,

$$\|z - sx\| = \frac{1}{1+s} = 1 - s + \frac{s^2}{1+s},$$

whence $z \in \mathrm{Lens}(\sqrt{2\delta}x, 2\delta(1+\sqrt{2\delta})^{-1})$ and (i) is proved. The inequalities (ii) and (iii) follow immediately from (i). □

It has to be noted that the three sets: the cap, the lens, and the drop remainder are similar with respect to property (β), and property CLUC. Alternative proofs of Propositions 2.5 and 2.6 can be obtained via results in [14], inequality (2.1) (Lemma 3.1 (ii), [21]), and Lemma 2.8.

3. Property (β) in terms of graphs

In this small section the property (β) is characterized by families of locally finite graphs. We shall use the following isometric characterization of property (β) which is a partial case of Theorem 7 from [13]:

Theorem 3.1 *([13]). A Banach space X has property (β) if, and only if, for each $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that for every $x \in B_X$ and every sequence $(x_n)_n \subset B_X$ with $\operatorname{sep}(x_n) > \varepsilon$ there is an index n_k such that $[x, x_{n_k}] \cap (1-\delta)B_X \neq \emptyset$.*

A graph Γ in a Banach space X is a pair of sets (V, E) called vertices and edges, respectively, such that V is a subset X and E is a set of unordered pairs of elements from V, i.e. we consider simple infinite undirected without loops graphs. It is convenient, for our purpose, to identify graphs with pairs (V, ϕ) where ϕ is an adjacency relation defining the set of edges, i.e. ϕ is a symmetric function on $V \times V$ with values 0 and 1. Thus, for $u, v \in V$, $\phi(u, v) = \phi(v, u) = 1$ means that u and v are connected by an edge, and the value of ϕ is 0 when they are not. Formally always, $\phi(u, u) = 0$ whenever $u \in V$.

The degree of a vertex $v \in V$ in a graph $\Gamma = (V, \phi)$ is the cardinality of the set of vertices in Γ that are connected to v, i.e. $\deg(v) = \operatorname{card}\{u \in V : \phi(u, v) = 1\}$.

Suppose, for $\varepsilon \in (0, 1)$, and $\delta \in (0, 1)$, a class of graphs $\mathsf{G}_\varepsilon^\delta$ is defined by the following construction: Let $V_\varepsilon^\delta \subset B \setminus (1-\delta)B$ be a set (of arbitrary possible cardinality) such that $\|u - v\| > \varepsilon$ whenever $u, v \in V_\varepsilon^\delta$ and $u \neq v$. Consider the graph $\Gamma(V_\varepsilon^\delta) = (V_\varepsilon^\delta, \phi_\delta)$ such that $\phi_\delta(u, v) = 1$ *iff* $[u, v] \cap (1-\delta)B = \emptyset$ for $u, v \in V_\varepsilon^\delta$. Then

$$\mathsf{G}_\varepsilon^\delta = \{\Gamma(V_\varepsilon^\delta) : \varepsilon \in (0, 1), \delta \in (0, 1)\}.$$

Proposition 3.2. *A Banach space X has property (β) if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $\Gamma \in \mathsf{G}_\varepsilon^\delta$ all vertices in Γ have finite degrees.*

Proof. Assume first, that the property from Theorem 3.1 is valid. For $\varepsilon > 0$ let $\delta > 0$ comply with Theorem 3.1. Let $\Gamma = (V_\varepsilon^\delta, \phi_\delta) \in \mathsf{G}_\varepsilon^\delta$. If a vertex $v \in V_\varepsilon^\delta$ exists with $\deg(v) \geq \aleph_0$ then there is a sequence (v_n) in B, such that $\|v_i - v_j\| > \varepsilon$, for all i, j, $i \neq j$, and $[v, v_n] \cap (1-\delta)B = \emptyset$ for every $n \in \mathbb{N}$, which is a contradiction.

Conversely, for $\varepsilon > 0$, let $\delta > 0$ be such that for every $\Gamma \in \mathsf{G}_{\varepsilon/2}^\delta$ all vertices of Γ have finite degrees. Let $x \in B$, and (x_n) is an ε-separated sequence in B, i.e. $\|x_i - x_j\| > \varepsilon$, $i, j \in \mathbb{N}, i \neq j$. Then after eventually removing an element from (x_n), say x_1, the sequence $\chi = (x, x_2, \ldots)$ is $\varepsilon/2$-separated. Indeed, if the sequence $(x, x_1, x_2, \ldots)$ is not so, assume with no loss of generality that $\|x - x_1\| \leq \varepsilon/2$. Then

$$\varepsilon < \|x_1 - x_n\| \leq \|x_1 - x\| + \|x - x_n\| \leq \varepsilon/2 + \|x - x_n\|,$$

whence $\|x - x_n\| > \varepsilon/2$ for $n > 1$. Thus χ is $\varepsilon/2$-separated.

Put $V = \{x, x_2, \ldots\}$. We may assume $V \subset B \setminus (1-\delta)B$. According to the choice of δ, the graph $\Gamma = (V, \phi_\delta)$ has all vertices of finite degrees. Since V is infinite, due to the separation property of its elements, there is n_k such that x and x_{n_k} are disconnected which means $[x, x_{n_k}] \cap (1-\delta)B \neq \emptyset$. □

4. Property (β) for $\ell_p \check{\otimes} \ell_q$ and $\ell_p \hat{\otimes} \ell_q$

Throughout this section $1 < p, q < \infty$. Let $\mathcal{K}(\ell_p, \ell_q)$ denote the space of all compact operators $T : \ell_p \to \ell_q$ with operator norm $\|T\|_\infty$, and let $\mathcal{N}(\ell_q, \ell_p)$ denote the space of nuclear operators $T = \sum_{n=1}^\infty x_n^* \otimes y_n$, where $(x_n^*) \subset \ell_q^* = \ell_{q'}$ $(1/q + 1/q' = 1)$ and $(y_n) \subset \ell_p$, equipped with the nuclear norm

$$\|T\|_1 = \inf\left\{\sum_{n=1}^\infty \|x_n^*\|_{q'}\|y_n\|_p : T = \sum_{n=1}^\infty x_n^* \otimes y_n\right\}.$$

It is well-known that the dual of $\mathcal{K}(\ell_p, \ell_q)$ is naturally isometrically isomorphic to $\mathcal{N}(\ell_q, \ell_p)$ under the duality pairing

$$(T, S) = \operatorname{trace}(ST) = \sum_{n=1}^\infty \sum_{m=1}^\infty x_n^*(Te_m)e_m^*(y_n),$$

where $(e_m)_{m=1}^\infty$ is the standard basis of ℓ_p and $(e_m^*)_{m=1}^\infty$ its dual basis. It is again well-known that $\mathcal{K}(\ell_p, \ell_q)$ is isometrically isomorphic to the projective tensor product $\ell_p \check{\otimes} \ell_{q'}$, while $\mathcal{N}(\ell_q, \ell_p)$ is isometrically isomorphic to the injective tensor product $\ell_{p'} \hat{\otimes} \ell_q$. We refer the reader to [6] for these facts and for further information about tensor products of Banach spaces.

In [7, Proposition 15] it was proved that $\mathcal{K}(\ell_p, \ell_q)$ always contains ℓ_∞^n's uniformly. By duality it follows that $\mathcal{N}(\ell_q, \ell_p)$ contains ℓ_1^n's uniformly. In particular, these spaces all fail to admit uniformly convex or uniformly smooth renormings. The main result of this section is that they nevertheless enjoy property (β) if (and only if) $p > 2 > q$.

Let τ be a Hausdorff topology on a Banach space X (usually the w- or w^*-topology). Let us recall the notions of *nearly uniformly smooth* and *uniformly Kadec–Klee* with respect to τ, denoted NUS(τ) and UKK(τ), and the moduli associated with them, introduced by Prus [19,20], and also the notion of *nearly uniformly convex*, denoted NUC, introduced by Huff [9] (Huff proved that his original definition of NUC is equivalent to what is stated below). These properties in combination with the results of Kutzarova [12] will be used to obtain property (β) in Theorem 4.9 below.

Definition 4.1. (a) For $t > 0$, let

$$b_{X,\tau}(t) = \sup\{\limsup_{n\to\infty} \|x + tx_n\| - 1\}$$

where the supremum is taken over all $x \in B_X$ and τ-null sequences $(x_n)_{n=1}^{\infty} \subset B_X$.
(b) X is NUS(τ) if $b_{X,\tau}(t) = o(t)$ as $t \to 0$.
(c) X is *nearly uniformly smooth*, denoted NUS, if X is NUS(w) and reflexive.
(d) For $t > 0$, let

$$d_{X,\tau}(t) = \inf\{\liminf_{n\to\infty} \|x + tx_n\| - 1\}$$

where the infimum is taken over all $x \in X$ with $\|x\| \geq 1$ and all τ-null sequences $(x_n)_{n=1}^{\infty}$ with $\|x_n\| \geq 1$.
(e) X is UKK(τ) if $d_{X,\tau}(t) > 0$ for all $t > 0$.
(f) X is NUC if X is UKK(w) and reflexive.

Example 4.2. Note that for $X = \ell_p$, we have $b_{X,w}(t) \asymp d_{X,w}(t) \asymp t^p$.

Theorem 4.3. *Let $X = \mathcal{K}(\ell_p, \ell_q)$. Then $b_{X,w}(t) \asymp t^r$ and $d_{X^*,w^*}(t) \asymp t^{r'}$, where $r = \min(p', q)$.*

Proof. Note that $(e_n^* \otimes e_1)_{n=1}^{\infty}$ and $(e_1^* \otimes e_n)_{n=1}^{\infty}$ are isometrically equivalent to the unit vector bases of $\ell_{p'}$ and ℓ_q respectively. Hence, by Example 4.2,

$$b_{X,w}(t) \geq ct^r \quad \text{for some } c > 0. \tag{4.1}$$

Similarly,

$$d_{X^*,w^*}(t) \leq ct^{r'} \quad \text{for some } c > 0. \tag{4.2}$$

Now we extend the argument of Besbes [3]. Let P denote one of the basis projections in ℓ_p, i.e.

$$P\left(\sum_{i=1}^{\infty} e_i^*(x)e_i\right) = \sum_{i=1}^{n} e_i^*(x)e_i \quad (x \in \ell_p),$$

where n is a fixed positive integer, let $Q = I - P$ be the complementary projection, and let $\tilde{P}$ be a basis projection in ℓ_q with complementary projection $\tilde{Q}$. Then, for all $T \in \mathcal{K}(\ell_p, \ell_q)$,

$$\begin{aligned}
\|T\|_\infty &\leq \sup_{a^p+b^p=1} (a\|TP\|_\infty + b\|TQ\|_\infty) \\
&= (\|TP\|_\infty^{p'} + \|TQ\|_\infty^{p'})^{1/p'} \\
&\leq ((\|\tilde{P}TP\|_\infty^{q} + \|\tilde{Q}TP\|_\infty^{q})^{p'/q} + \|TQ\|_\infty^{p'})^{1/p'} \\
&\leq (\|\tilde{P}TP\|_\infty^{r} + \|\tilde{Q}TP\|_\infty^{r} + \|TQ\|_\infty^{r})^{1/r}.
\end{aligned} \tag{4.3}$$

Now suppose that $T \in B_X$ and that $(T_n)_{n=1}^{\infty} \subset B_X$ is weakly null. Hence

$$\limsup_{n\to\infty} \|T + tT_n\|_\infty \leq \limsup_{n\to\infty} (\|\tilde{P}(T + tT_n)P\|_\infty^r + \|\tilde{Q}(T + tT_n)P\|_\infty^r + \|(T + tT_n)Q\|_\infty^r)^{1/r}.$$

We have $\|\tilde{P}T_nP\|_\infty \to 0$ as $n \to \infty$, and $\|T - \tilde{P}TP\|_\infty \to 0$ as $\min(\operatorname{rank}(P), \operatorname{rank}(\tilde{P})) \to \infty$. Hence, taking the limit as $\min(\operatorname{rank}(P), \operatorname{rank}(\tilde{P})) \to \infty$, we get

$$\limsup_{n\to\infty} \|T + tT_n\|_\infty \leq (\|T\|_\infty^r + 2t^r \limsup_{n\to\infty} \|T_n\|_\infty^r)^{1/r} \leq (1 + 2t^r)^{1/r}. \tag{4.4}$$

Combining (4.1) and (4.4) yields $b(t) \asymp t^r$. Dualizing (4.3) yields for $T \in X^* = \mathcal{N}(\ell_q, \ell_p)$

$$\|T\|_1 \geq (\|PT\tilde{P}\|_1^{r'} + \|PT\tilde{Q}\|_1^{r'} + \|QT\|_1^{r'})^{1/r'}. \tag{4.5}$$

Suppose $\|T\|_1 \geq 1$, $\|T_n\|_1 \geq 1$, and that $(T_n)_{n=1}^{\infty}$ is w^*-null. Then (4.5) yields

$$\liminf_{n\to\infty} \|T + tT_n\|_1 \geq \liminf_{n\to\infty} (\|P(T + tT_n)\tilde{P}\|_1^{r'} + \|P(T + tT_n)\tilde{Q}\|_1^{r'} + \|Q(T + tT_n)\|_1^{r'})^{1/r'}. \tag{4.6}$$

Now $\|PT_n\tilde{P}\|_1 \to 0$ as $n \to \infty$, and both $\|PT\tilde{Q}\|_1 \to 0$ and $\|QT\|_1 \to 0$ as $\min(\operatorname{rank}(P), \operatorname{rank}(\tilde{P})) \to \infty$. So, given $\varepsilon > 0$, the triangle inequality yields

$$\liminf_{n\to\infty} (\|P(T + tT_n)\tilde{Q}\|_1 + \|Q(T + tT_n)\|_1) \geq (1 - \varepsilon)t$$

provided $\min(\operatorname{rank}(P), \operatorname{rank}(\tilde{P}))$ is sufficiently large. Hence (4.6) yields

$$\liminf_{n\to\infty} \|T + tT_n\|_1 \geq \liminf_{n\to\infty} \left(1 + \left(\frac{t\|T_n\|}{2}\right)^{r'}\right)^{1/r'} \geq \left(1 + \left(\frac{t}{2}\right)^{r'}\right)^{1/r'}. \tag{4.7}$$

Combining (4.2) and (4.7) yields $d_{X^*,w^*}(t) \asymp t^{r'}$. □

Corollary 4.4. *Let $1 < q < p < \infty$. Then $\mathcal{K}(\ell_p, \ell_q)$ is NUS and $\mathcal{N}(\ell_q, \ell_p)$ is NUC.*

Proof. Recall that $\mathcal{K}(\ell_p, \ell_q)$ is reflexive if and only if $1 < q < p < \infty$, so this is also a necessary condition for the NUC property. On the other hand, by Theorem 4.3, $\mathcal{K}(\ell_p, \ell_q)$ is NUS(w) for all $1 < p, q < \infty$, which establishes sufficiency. The statement for $\mathcal{N}(\ell_q, \ell_p)$ follows from the duality between NUC and NUS [19]. □

Van Dulst and Sims [26] proved that the UKK(w^*) property for a dual space X^* implies the w^*-fixed point property, i.e., that every nonexpansive self-mapping of a w^*-compact convex subset of X^* has a fixed point. Hence we obtain the following application of Theorem 4.3 to fixed point theory. The special case $p = q = 2$ is the "trace class" $\mathcal{C}_1$, which was proved by Lennard [15], and the case $q = p'$ was proved by Besbes [3].

Theorem 4.5. *let $1 < p, q < \infty$. Then $\mathcal{N}(\ell_q, \ell_p)$ has the w^*-fixed point property.*

Theorem 4.6. *Let $p > 2 > q$. Then $X = \mathcal{K}(\ell_p, \ell_q)$ is NUC and $d_{X,w}(t) \asymp t^r$, where*

$$r = \max\left(\frac{2p}{p-2}, \frac{2q}{2-q}\right).$$

Proof. It was proved in [7, Theorem 4] that $\mathcal{K}(\ell_p, \ell_q)$ is NUC in this range and that $d_{X,w}(t) \geq ct^r$. Hence it suffices to show that $d_{X,w}(t) \leq ct^r$ for some $c > 0$. To that end, consider $T = 2^{-1/q}e_1^* \otimes (e_1 + e_2)$ and, for $n \geq 1$, $T_n = 2^{-1/q}e_n^* \otimes (e_1 - e_2)$. Then $\|T\|_\infty = \|T_n\|_\infty = 1$ and $(T_n)_{n=1}^\infty$ is weakly null. Note that, for $t > 0$,

$$\begin{aligned}\|T + tT_n\|^q &= \frac{1}{2} \max_{0\leq x\leq 1} ((1 - x^p)^{1/p} + xt)^q + ((1 - x^p)^{1/p} - xt)^q \\ &= \max_{0\leq x\leq 1} \left(1 - \frac{q}{p}x^p + \frac{q(q-1)}{2}x^2t^2 + \text{smaller terms}\right).\end{aligned}$$

The maximum value of $1 - \frac{q}{p}x^p + \frac{q(q-1)}{2}x^2t^2$ as x ranges over $[0, 1]$ is given by

$$1 + \frac{q(p-2)}{2p}(q-1)^{p/(p-2)}t^{2p/(p-2)}.$$

Hence $\|T + tT_n\|_\infty \leq 1 + ct^{2p/(p-2)}$ for some $c > 0$, which gives $d_{X,w}(t) \leq ct^{2p/(p-2)}$. But $\mathcal{K}(\ell_p, \ell_q)$ is isometrically isomorphic to $\mathcal{K}(\ell_{q'}, \ell_{p'})$ via the mapping $T \mapsto T^*$, where T^* is the adjoint of T. So we also have $d_{X,w}(t) \leq ct^{2q'/(q'-2)}$. But $q'/(q'-2) = q/(2-q)$, so $d_{X,w}(t) \leq ct^r$ as desired. □

Corollary 4.7. *Let $p > 2 > q$. Then $X = \mathcal{N}(\ell_q, \ell_p)$ is NUS and $b_{X,w}(t) \asymp t^s$, where $s = \min\left(\frac{2p}{p+2}, \frac{2q}{3q-2}\right)$.*

Proof. Note that $s = r'$, where r is as in Theorem 4.6. Now $\mathcal{N}(\ell_q, \ell_p)$ is reflexive in this range with dual $\mathcal{K}(\ell_p, \ell_q)$, so the result follows from Theorem 4.6 and the duality formula relating $b_{X,w}(t)$ and $d_{X^*,w}(t)$ in reflexive spaces proved in [20, Theorem 4.17]. □

Remark 4.8. It was proved in [7, Theorem 13] that for the complementary range, i.e., if $p \leq q$ or $1 < q < p \leq 2$ or $2 \leq q < p$, then $\mathcal{K}(\ell_p, \ell_q)$ does not admit an *equivalent norm* with the UKK property. In particular, $\mathcal{K}(\ell_p, \ell_q)$ cannot be renormed to be NUC. By duality it follows that $\mathcal{N}(\ell_q, \ell_p)$ cannot be renormed to be NUS in this range.

Theorem 4.9. (1) *If $1 < q < 2 < p < \infty$ then $\mathcal{K}(\ell_p, \ell_q)$ and $\mathcal{N}(\ell_q, \ell_p)$ have property (β).*
(2) *For the complementary range of values of p and q, neither $\mathcal{K}(\ell_p, \ell_q)$ nor $\mathcal{N}(\ell_q, \ell_p)$ admits an equivalent norm with property (β).*

Proof. (1) By Theorems 4.3 and 4.6, $1 < q < 2 < p < \infty$ is the range for which $\mathcal{K}(\ell_p, \ell_q)$ and $\mathcal{N}(\ell_q, \ell_p)$ are simultaneously NUC and NUS. By a result of Kutzarova [12], NUC and NUS together imply property (β).

(2) It is known that property (β) implies NUC [12]. Hence by Remark 4.8, $\mathcal{K}(\ell_p, \ell_q)$ cannot be renormed to have property (β). By [12] a Banach space with a Schauder basis which enjoys property (β) admits an NUS renorming. Hence, by Remark 4.8, $\mathcal{N}(\ell_q, \ell_p)$ cannot be renormed to have property (β). □

Finally, we translate Theorem 4.9 into the language of tensor products.

Corollary 4.10. (1) *The projective tensor product* $\ell_p\check{\otimes}\ell_q$ *has property* (β) *if and only if* $\min(p, q) > 2$.
(2) *The injective tensor product* $\ell_p\hat{\otimes}\ell_q$ *has property* (β) *if and only if* $\max(p, q) < 2$.

5. (β)-modulus

We shall give an estimate of the (β)-modulus if we are given estimates for the moduli $b_X(t)$ and $d_X(t)$, which were defined in Definition 4.1 and τ is the weak topology of X. Recall that the (β)-modulus [1] is given by

$$\overline{\beta}_X(t) = 1 - \sup\left\{\inf\left\{\frac{\|x+x_n\|}{2} : n \in \mathbb{N}\right\} : (x_n)_n \subset B_X, x \in B_X, \operatorname{sep}((x_n)_n) \geq t\right\}.$$

Lemma 5.1. *Assume that X has property (β). Let $\varepsilon > 0$, and let $x, x_n \in B_X$ be such that $\overline{\beta}_X(\varepsilon)$ is approximated by*

$$1 - \inf\left\{\frac{\|x+x_n\|}{2}, n \geq 1\right\}.$$

Call $\sigma := \operatorname{sep}((x_n)_n)$. Then the choice of x and x_n can be made so that $\sigma \to 0$ as $\varepsilon \to 0$.

Proof. First we show that $\overline{\beta}_X(\varepsilon) \to 0$ as $\varepsilon \to 0$ for any infinite-dimensional Banach space X. To achieve this, take $\|y\| = 1 - \varepsilon$ and $y_n = y + \varepsilon u_n$, where $(u_n)_n$ is a sequence of unit vectors with $\|u_n - u_m\| \geq 1$. (The existence of such a sequence $(u_n)_n$ follows from the proof of Mazur's Lemma.) Then $y, y_n \in B_X$ and $\|y + y_n\|/2 \geq 1 - (3/2)\varepsilon$, and hence $\overline{\beta}_X(\varepsilon) \leq (3/2)\varepsilon$.

Now when X has property (β), this together with the fact that $\overline{\beta}_X(\varepsilon) > 0$ for $\varepsilon > 0$ implies that there exist two sequences $(\varepsilon_k)_k \downarrow 0$ and $(\delta_k)_k \downarrow 0$ such that $\overline{\beta}_X(\varepsilon_k) < \overline{\beta}_X(\varepsilon_{k-1})$ and $[t < \delta_k \Rightarrow \overline{\beta}_X(t) < \overline{\beta}_X(\varepsilon_k)]$. Call $\alpha_k = \left(\overline{\beta}_X(\varepsilon_{k-1}) - \overline{\beta}_X(\varepsilon_k)\right)/2$.

Let $\varepsilon > 0$ be small enough and assume $\varepsilon < \delta_k$. Let $\|x\| \leq 1$, $\|x_n\| \leq 1$, with $\sigma := \operatorname{sep}((x_n)_n) \geq \varepsilon$, be such that

$$\inf\left\{\frac{\|x+x_n\|}{2}, n \geq 1\right\} > 1 - \overline{\beta}_X(\varepsilon) - \alpha_k.$$

We claim that we can always make such a choice of x and $(x_n)_n$ such that $\sigma < \varepsilon_{k-1}$. In fact, otherwise we would have $\sigma \geq \varepsilon_{k-1}$. This would give

$$\begin{aligned}
1 - \overline{\beta}_X(\varepsilon_{k-1}) &\geq 1 - \overline{\beta}_X(\sigma) \\
&\geq \inf\left\{\frac{\|x+x_n\|}{2}, n \geq 1\right\} \\
&> 1 - \overline{\beta}_X(\varepsilon) - \alpha_k \\
&> 1 - \overline{\beta}_X(\varepsilon_k) - \alpha_k.
\end{aligned}$$

The first inequality is because $\overline{\beta}_X(\cdot)$ is nondecreasing. The last inequality is because $\varepsilon < \delta_k$. All these imply

$$\overline{\beta}_X(\varepsilon_k) + \alpha_k > \overline{\beta}_X(\varepsilon_{k-1}),$$

which contradicts the definition of α_k. □

Theorem 5.2. *Assume that there are constants $D > 0$, $B > 0$, $1 < p \leq q < \infty$ such that $b_X(t) \leq Bt^p$ and $d_X(t) \geq Dt^q$. Then $\overline{\beta}_X(t) \geq Kt^s$ for some constant $K > 0$, where $s = \frac{qp-p}{p-1}$.*

Proof. Let $\varepsilon > 0$. Let $\|x\| \leq 1$, $\|x_n\| \leq 1$ with $\operatorname{sep}((x_n)_n) \geq \varepsilon$ be so that $1-\overline{\beta}_X(\varepsilon)$ is approximated by $\inf\{\|x+x_n\|/2, n \geq 1\}$. We already know that X has property (β), so it must be reflexive. Hence by taking a subsequence let us assume that $x_n \to u$ weakly for some $u \in B_X$. Also, by taking a further subsequence, let us assume that $\lim_{n\to\infty} \|x_n - u\| = t$ for some $t > 0$.

Note that since $\varepsilon \leq \|x_n - x_m\| \leq \|x_n - u\| + \|x_m - u\|$, we must have $t \geq \varepsilon/2$. Also, by using Lemma 5.1, we can assume that $t \to 0$ as $\varepsilon \to 0$.

We will show that $\|u\| \leq 1 - \frac{D}{2}t^q$. This is definitely true if $u = 0$, so for the proof we assume that $u \neq 0$. We write

$$\frac{x_n}{\|u\|} = \frac{u}{\|u\|} + \frac{x_n - u}{\|u\|}.$$

Note that $\left(\frac{x_n-u}{\|u\|}\right)_n$ is weakly null, and as $n \to \infty$, we have for any $\alpha > 1$:

$$\frac{\|x_n - u\|}{\|u\|} \geq \|x_n - u\| > \frac{t}{\alpha}.$$

So by the definition of the modulus $d_X(\cdot)$, we have

$$\begin{aligned} 1+d_X(t/\alpha) &\le \liminf \left\| \frac{u}{\|u\|} + \frac{x_n-u}{\|u\|} \right\| \\ &= \liminf \frac{\|x_n\|}{\|u\|} \\ &\le \frac{1}{\|u\|}. \end{aligned}$$

By the estimate we have on $d_X(\cdot)$, we then have

$$\|u\| \le \frac{1}{1+Dt^q/\alpha^q}.$$

Then since $\alpha > 1$ is arbitrary, $\|u\| \le \frac{1}{1+Dt^q} \le 1 - \frac{D}{2}t^q$ if t is small enough.

Next, consider $0 < \lambda < 1$. We have

$$\|(1-\lambda)x + \lambda u\| \le (1-\lambda) + \lambda\left(1 - \frac{D}{2}t^q\right) = 1 - \frac{D}{2}\lambda t^q.$$

Hence the vector $y := \frac{(1-\lambda)x+\lambda u}{1-\frac{D}{2}\lambda t^q}$ belongs to the unit ball. We then have

$$\begin{aligned} \limsup \|(1-\lambda)x + \lambda x_n\| &= \left(1 - \frac{D}{2}\lambda t^q\right) \limsup \left\| y + \frac{2^{1/p} t\lambda}{1-\frac{D}{2}\lambda t^q} \cdot \frac{x_n - u}{2^{1/p}t} \right\| \\ &\le \left(1 - \frac{D}{2}\lambda t^q\right)\left(1 + b_X\left(\frac{2^{1/p}t\lambda}{1-\frac{D}{2}\lambda t^q}\right)\right) \quad \text{since } \left\|\frac{x_n-u}{2^{1/p}t}\right\| \le 1 \\ &\le \left(1 - \frac{D}{2}\lambda t^q\right)\left(1 + B\cdot\frac{2t^p\lambda^p}{\left(1-\frac{D}{2}\lambda t^q\right)^p}\right) \\ &\le \left(1 - \frac{D}{2}\lambda t^q\right)\left(1 + 4Bt^p\lambda^p\right) \quad \text{if } t \text{ is small enough.} \end{aligned}$$

Hence we can find an index n_0 (depending on λ) such that

$$\begin{aligned} \|(1-\lambda)x + \lambda x_{n_0}\| &\le \left(1 - \frac{D}{2}\lambda t^q\right)(1 + 8Bt^p\lambda^p) \\ &= 1 - \frac{D}{2}\lambda t^q + 8B\lambda^p t^p - 4DB\lambda^{1+p}t^{p+q}. \end{aligned}$$

This is true for any $0 < \lambda < 1$. Let us require that $8B\lambda^p t^p = \frac{D}{4}\lambda t^q$. To do this, we consider two cases. If $q = p$, we set $\lambda = \left(\frac{D}{32B}\right)^{1/(p-1)}$, and note that this particular λ is less than 1 since $D \le B$ when $p = q$. For the case $p < q$, we set $\lambda = \left(\frac{D}{32B}t^{q-p}\right)^{1/(p-1)}$, and note that such λ is less than 1 if t is small enough. With these respective choices of λ we have

$$\|(1-\lambda)x + \lambda x_{n_0}\| \le 1 - K_1 t^s - K_2 t^{2s} \le 1 - K_1 t^s,$$

where $s = \frac{qp-p}{p-1}$.

Now,

$$\begin{aligned} \left\|\frac{x + x_{n_0}}{2}\right\| &= \left\| \frac{1}{2(1-\lambda)}[(1-\lambda)x + \lambda x_{n_0}] + \frac{1-2\lambda}{2(1-\lambda)}x_{n_0} \right\| \\ &\le \frac{1}{2(1-\lambda)}(1 - K_1 t^s) + \frac{1-2\lambda}{2(1-\lambda)} \\ &= 1 - \frac{K_1 t^s}{2(1-\lambda)} \\ &\le 1 - \frac{K_1}{2}t^s. \end{aligned}$$

Note that K_1 depends only on D, B, p and q. Since $t \ge \varepsilon/2$, the definition of the (β)-modulus $\overline{\beta}_X(\cdot)$ then gives

$$\overline{\beta}_X(\varepsilon) \ge \frac{K_1}{2}t^s \ge K\varepsilon^s. \quad \square$$

We remark that our estimate is not optimal when $p < q$. In fact, if $X = L_r[0,1]$ with $1 < r < \infty$, then $p = \min(r, 2)$ and $q = \max(r, 2)$. One can check that $\overline{\beta}_X(t) \asymp t^q$. However,

$$s = \frac{qp - p}{p - 1} = \begin{cases} \dfrac{r}{r-1} > 2 & \text{for } r < 2, \\ 2(r-1) > r & \text{for } r > 2. \end{cases}$$

However, we show that our computation does give the optimal estimate when $q = p$.

Theorem 5.3. *Under the assumptions of Theorem 5.2, assume further that $p = q$. Then $Kt^q \le \overline{\beta}_X(t) \le K't^q$ for some constants $K, K' > 0$.*

Proof. Theorem 5.2 gives us $\overline{\beta}_X(t) \ge Kt^p$ since $s = \frac{qp-p}{p-1} = p$ when $q = p$.

For the reverse inequality, we start with the fact that $b_X(t) \le Bt^q$. Consider vectors v and v_n such that $\|v\| = 1$, $\|v_n\| \le 1$, $v_n \to 0$ weakly, and $\|v_n - v_m\| \ge 1/2$. (For example, one can take $v_n = (u_n - u)/2$ where $(u_n)_n$ is a subsequence of a sequence $(u_n)_n$ as at the beginning of the proof of Lemma 5.1, a subsequence that is weakly convergent to some vector u. Recall that a space with property (β) is reflexive.) By the definition of $b_X(\cdot)$, we have

$$\limsup \|v + tv_n\| \le 1 + Bt^q.$$

So we can extract a subsequence, still denoted $(v_n)_n$, such that $\|v + tv_n\| \le 1 + 2Bt^q$ for all n.

Call $x_n := \frac{1}{1+2Bt^q}(v + tv_n)$ and $x := \frac{1}{1+2Bt^q}v$. We have $x_n \in B_X$, and hence x, which is the weak limit of $(x_n)_n$, is also in B_X. We also have

$$\|x_n - x_m\| = \frac{t}{1 + 2Bt^q}\|v_n - v_m\| \ge \frac{t}{2 + 4Bt^q} \ge \frac{1}{3}t.$$

Hence,

$$\liminf \left\| \frac{x + x_n}{2} \right\| \le 1 - \overline{\beta}_X(t/3).$$

On the other hand, since x is the weak limit of $(x + x_n)/2$, we have

$$\begin{aligned} \liminf \left\| \frac{x + x_n}{2} \right\| &\ge \|x\| \\ &= \frac{1}{1 + 2Bt^q}\|v\| \\ &= \frac{1}{1 + 2Bt^q} \\ &\ge 1 - 2Bt^q. \end{aligned}$$

This gives that $\overline{\beta}_X(t/3) \le 2Bt^q$. □

Remark. In Theorem 4.9 we established the range of the parameters p and q for which the spaces $\mathcal{K}(\ell_p, \ell_q)$ and $\mathcal{N}(\ell_q, \ell_p)$ have property (β), by evaluating the power types of their NUC and NUS moduli. Therefore, we can directly apply Theorem 5.2 to obtain some estimates for the (β)-modulus of these spaces. We do not know their exact power types.

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