THE TRANSFER OF PROPERTY (β) OF ROLEWICZ BY A UNIFORM QUOTIENT MAP

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Abstract. We provide a Laakso construction to prove that the property of having an equivalent norm with the property (β) of Rolewicz is qualitatively preserved via surjective uniform quotient mappings between separable Banach spaces. On the other hand, we show that the (β)-modulus is not quantitatively preserved via such a map by exhibiting two uniformly homeomorphic Banach spaces that do not have (β)-moduli of the same power-type even under renorming.

1. Introduction

A map $f : X \longrightarrow Y$ between two metric spaces is called a uniform quotient mapping [5, Chapter 11] if there exist two nondecreasing maps $\omega, \Omega : [0, \infty) \longrightarrow [0, \infty)$ such that $\omega(r) > 0$ for $r > 0$, and $\Omega(r) \to 0$ as $r \to 0$, and for every $x \in X$ and every $r > 0$,

$$B(f(x), \omega(r)) \subseteq f(B(x, r)) \subseteq B(f(x), \Omega(r)).$$

(In this article, all balls will be considered closed unless otherwise specified.) If there are constants $c, L > 0$ such that one can use $\omega(r) = cr$ and $\Omega(r) = Lr$, then the map f is called a Lipschitz quotient mapping. If the map f is surjective, the metric space Y is called a uniform quotient (resp. Lipschitz quotient) of the metric space X.

Closely related to the above definitions are the notions of coarse Lipschitz maps and coarse co-Lipschitz maps. A map $f : X \to Y$ between two metric spaces is called Lipschitz for large distances, or coarse Lipschitz, if for every $d > 0$ there exists $L(d) > 0$ such that $\operatorname{dist}(f(s), f(s')) \leq L(d) \operatorname{dist}(s, s')$ for every $s, s' \in X$ with $\operatorname{dist}(s, s') \geq d$. The map f is called co-Lipschitz for large distances, or coarse co-Lipschitz, if for every $d > 0$ there exists $c(d) > 0$ such that for every $R \geq d$, for every $x, y \in Y$ with $\operatorname{dist}(x, y) \leq c(d)R$, for every $s \in f^{-1}(x)$, there exists $s' \in f^{-1}(y)$ with $\operatorname{dist}(s, s') \leq R$. It is known that when the metric spaces X and Y are metrically convex (for example if X and Y are Banach spaces), then a surjective uniform quotient mapping between X and Y is always coarse Lispchitz and coarse co-Lipschitz (see [5, Chapter 11] and [24, Section 3] for example).

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In [24] and in [9] the geometric property (β) of Rolewicz is applied to the study of the rigidity of Banach spaces under uniform quotient mappings. Rolewicz introduced property (β) as an isometric generalization of uniform convexity in the sense that a uniformly convex Banach space necessarily has property (β) (see [30] and [31]). Furthermore a result of Baudier, Kalton, and Lancien [4] implies that a separable reflexive Banach space X can be renormed to have property (β) if and only if the infinitely branching infinitely deep tree with its hyperbolic metric cannot bi-Lipschitz embed into X. Contrasted with Bourgain's metric characterization of superreflexivity (see [6] and [3]), this indicates that for separable spaces the isomorphic class of all Banach spaces with property (β) is also larger than the isomorphic class of all uniformly convex Banach spaces. We remark in passing that this statement is actually known to be true, see for example [19].

For the purpose of this article, the equivalent definition of property (β) given in [21] is used. A Banach space X has property (β) if for any $t \in (0, a]$, where the number $1 \leq a \leq 2$ depends on the space X, there exists $\delta > 0$ such that for any x in the unit ball B_X, and any t-separated sequence $(x_n)_{n=1}^{\infty}$ in B_X, there exists $n \geq 1$ satisfying

$$\frac{\|x - x_n\|}{2} \leq 1 - \delta.$$

The (β)-modulus was defined in [1] (although with a different notation) as

$$\overline{\beta}_X(t) = 1 - \sup\left\{\inf\left\{\frac{\|x - x_n\|}{2} : n \in \mathbb{N}\right\} : (x_n)_n \subseteq B_X, x \in B_X, \operatorname{sep}((x_n)_n) \geq t\right\},$$

where $\operatorname{sep}((x_n)_{n\geq 1}) = \inf\{\|x_n - x_m\| : n \neq m, n, m \geq 1\}$.

In [9] it is proven that if a Banach space Y is a uniform quotient of another Banach space X with property (β), then one can compare the (β)-modulus of X with the modulus of another asymptotic geometry on Y, namely asymptotic uniform smoothness, and under additional assumptions, asymptotic uniform convexity. (See [26] and [15] for the notion of asymptotic uniform smoothness and asymptotic uniform convexity. See also section 4.) On the other hand, the authors in [25] use the (local) geometric property called Markov convexity to study the rigidity of Banach spaces under Lipschitz quotient mappings. They do this by comparing the Markov convexity of a space Y with the Markov convexity of a space X when Y is a Lipschitz quotient of a subset of X.

In light of these two approaches, the present article studies the asymptotic property (β) of Rolewicz on a Banach space Y that is a uniform quotient of another Banach space X with property (β). This is done both in a qualitative sense and a quantitative sense. We show that the property of having an equivalent norm with property (β) is qualitatively preserved under a surjective uniform quotient mapping between separable Banach spaces. We do this by first assuming reflexivity on the range space in section 2. In section 3, we provide a modification of this approach to remove the reflexivity assumption.

In contrast, we show in section 4 that quantitatively the modulus of (β) is not preserved under a surjective uniform quotient mapping. We also show that in full generality we cannot compare the (β)-modulus of the domain space with the modulus of asymptotic uniform convexity of the range space of a surjective uniform quotient map.

2. Case when the range space is reflexive

One of the main ingredients in this section is a metric characterization of being isomorphic to a space with property (β). By combining the results of [9, Theorem 6.3] and [4, Theorem 1.2], we know that a separable reflexive Banach space Y has an equivalent norm with property (β) if and only if it does not contain any bi-Lipschitz copy of the metric tree $\mathbb{T}_\infty$ of all finite subsets of $\mathbb{N}$ with the shortest path metric.

Theorem 2.0.1. *Let X be a Banach space with (an equivalent norm with) property (β). Let Y be a separable Banach space that is a uniform quotient of X. If Y is reflexive, then Y has an equivalent norm with property (β).*

Proof. Let $f : X \longrightarrow Y$ be a surjective uniform quotient mapping. Then f is coarse Lipschitz and coarse co-Lipschitz. For a contradiction, assume that Y does not admit any equivalent norm with property (β). Denote by Σ a subset of Y that is bi-Lipschitz equivalent to $\mathbb{T}_\infty$, with a bi-Lipschitz equivalence denoted by $j : \Sigma \longrightarrow \mathbb{T}_\infty$. The restriction $f_{|_{f^{-1}(\Sigma)}} : f^{-1}(\Sigma) \longrightarrow \Sigma$ is still a uniform quotient map that is coarse Lipschitz and coarse co-Lipschitz. As a result, since $\mathbb{T}_\infty$ is discrete, the composition $j \circ f_{|_{f^{-1}(\Sigma)}} : f^{-1}(\Sigma) \longrightarrow \mathbb{T}_\infty$ is a (surjective) Lipschitz quotient map. The theorem then follows from the next proposition.

□

Proposition 2.0.2. *It is not possible that $\mathbb{T}_\infty$ be a Lipschitz quotient of a subset of a Banach space with property (β).*

Proof. We define a graph $\mathbb{M}_\infty$ considered as a metric space, and we define a map $\phi : \mathbb{T}_\infty \longrightarrow \mathbb{M}_\infty$. This map ϕ, although not a Lipschitz quotient map, will have a property still strong enough to draw a contradiction from the composition $\lambda = \phi \circ j \circ f_{|_{f^{-1}(\Sigma)}} : f^{-1}(\Sigma) \longrightarrow \mathbb{M}_\infty$, where $j \circ f_{|_{f^{-1}(\Sigma)}} : f^{-1}(\Sigma) \longrightarrow \mathbb{T}_\infty$ is a Lipschitz quotient map from a subset $f^{-1}(\Sigma)$ of a Banach space with property (β). We support all these claims in the foregoing subsections. □

2.1. Definition of the graph $\mathbb{M}_\infty$.

The graph $\mathbb{M}_\infty$ is defined after a variant of the Laakso construction (see for example [8, 23, 25]). We define a sequence of graphs $(G_n)_{n\geq 1}$ such that $G_n \subseteq G_{n+1}$ for every $n \geq 1$ as follows. The graph G_1 is the following ordered graph. We start with a root. The root has one immediate descendant. This first descendant has countably infinitely many immediate descendants. Each one of those second generation vertices has only one immediate descendant, which is a common immediate descendant to all of them. The graph G_1 has diameter 3.

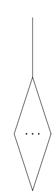

Figure 1. The graph G_1. The "$\cdots$" represents the fact that G_1 has infinitely many vertices at level 2.

To construct G_{n+1} ($n \geq 1$), we take a copy of G_1 and we rescale it so each edge has length the diameter of G_n. Next we replace each rescaled edge with a copy of G_n, matching the parent vertex of the rescaled edge with the oldest ancestor in the copy of G_n, and the child vertex of the rescaled edge with the youngest vertex in the copy of G_n. This gives all the vertices of G_{n+1}. Next we order the vertices of G_{n+1} by going downward from the root all the way to the youngest generation, which is now at distance 3^{n+1} from the root. Finally, we identify the root and the first 3^n generations of G_{n+1} with (the root and the first 3^n generations of) G_n. This way we have $G_n \subseteq G_{n+1}$.

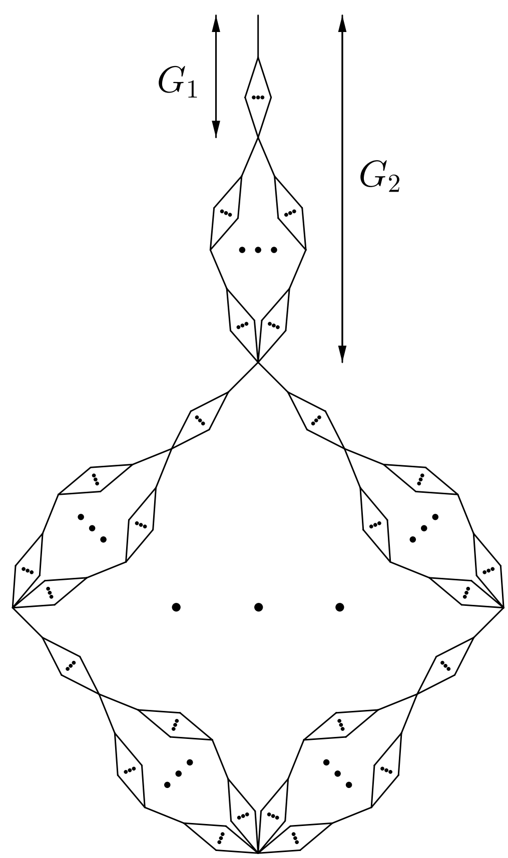

FIGURE 2. The graph G_3 with the subsets G_1 and G_2 highlighted.

The graph $\mathbb{M}_\infty$ is defined to be the union $\bigcup_{n=1}^{\infty} G_n$, considered as a metric space with the shortest path metric. The generational order of a given element of $\mathbb{M}_\infty$ will be called its *level*. Note that the level of an element is also the length of a shortest path from the root to that element. An element y is called a descendant of another element x, denoted $x < y$, if there is a shortest path from the root to y that passes through x.

An element of $\mathbb{M}_\infty$ will be called *non-branching* if it only has one immediate descendant; and it will be called *branching* if it has infinitely many immediate descendants. For each branching element of $\mathbb{M}_\infty$, we give a fraternal order among its immediate descendants by fixing a bijection between these descendants and the set $\mathbb{N}$ of the natural numbers.

Remark 2.1.1. Laakso constructions, introduced in [22], have appeared elsewhere as a tool to investigate the geometry of metric spaces, including that of Banach spaces. The following list is not exhaustive: [32], [16], [25], [8], and recently [28]. We would like to point out the difference between our Laakso construction and that of [25, Section 3] and [8, Example 1.2] for example. First, our basic block G_1 has diameter three instead of four. Second, in the construction of our basic block G_1, the first generation vertex has infinitely many immediate descendants instead of just two as in the standard Laakso construction. Third, when constructing G_{n+1} from G_n, we do *not* rescale down the copies of G_n. In our case, G_{n+1} has diameter three times as large as the diameter of G_n. We also note that even if the Laakso construction in [23] can be made so that the basic block has diameter three, our construction is still different from the construction in [23].

On the other hand we would like to mention that at the inception of this paper, our Laakso construction did have a basic block with diameter four. The proof went all the way through. However, as we prepared the final version of this article we realized that we did not need the last edge of the basic block, which brought us to the construction of $\mathbb{M}_\infty$ as it is presented here.

2.2. Definition and properties of the map $\phi : \mathbb{T}_\infty \longrightarrow \mathbb{M}_\infty$.

We define $\phi(\text{root}) = \text{root}$. By induction, if the $\phi(\overline{n})$ is already defined for an element $\overline{n}$ of $\mathbb{T}_\infty$, and if $\overline{m}$ is an immediate descendant of $\overline{n}$ in $\mathbb{T}_\infty$, then

- set $\phi(\overline{m})$ to be the one immediate descendant of $\phi(\overline{n})$ if $\phi(\overline{n})$ is non-branching;
- otherwise, set $\phi(\overline{m})$ to be the immediate descendant of $\phi(n)$ in the same fraternal order among all immediate descendants of $\phi(\overline{n})$ as $\overline{m}$ is among all immediate descendants of $\overline{n}$.

We list the properties of the map ϕ in the following lemma.

Lemma 2.2.1.

(1) *ϕ is surjective, and preserves levels.*
(2) *ϕ is 1-Lipschitz.*
(3) *ϕ is ancestor-to-descendant 1-co-Lipschitz in the following sense:*

$$\forall x, y \in \mathbb{M}_\infty \textit{ with } x < y,\ \forall \overline{n} \in \phi^{-1}(x),\ \exists \overline{m} \in \phi^{-1}(y) \textit{ such that } \operatorname{dist}_{\mathbb{T}_\infty}(\overline{n}, \overline{m}) = \operatorname{dist}_{\mathbb{M}_\infty}(x, y).$$

Proof.

(1) This claim is easy.
(2) Let $\overline{n}, \overline{m} \in \mathbb{T}_\infty$. We consider two cases.
    - Case $\overline{n} < \overline{m}$:

$$\begin{aligned}\operatorname{dist}_{\mathbb{M}_\infty}(\phi(\overline{n}),\phi(\overline{m})) &= \operatorname{level}(\phi(\overline{m})) - \operatorname{level}(\phi(\overline{n}))\\ &= \operatorname{level}(\overline{m}) - \operatorname{level}(\overline{n})\\ &= \operatorname{dist}_{\mathbb{T}_\infty}(\overline{n},\overline{m}).\end{aligned}$$

- Case $\overline{n}$ and $\overline{m}$ are not comparable:
  Let $\overline{o} \in \mathbb{T}_\infty$ be the closest common ancestor to $\overline{n}$ and $\overline{m}$ in $\mathbb{T}_\infty$; and let $x \in \mathbb{M}_\infty$ be the closest common ancestor to $\phi(\overline{n})$ and $\phi(\overline{m})$ in $\mathbb{M}_\infty$. Since $\phi(\overline{o})$ is also a common ancestor to $\phi(\overline{n})$ and $\phi(\overline{m})$, the definition of x gives us that $\operatorname{level}(\phi(\overline{o})) \leq \operatorname{level}(x)$. Hence we have:

$$\begin{aligned}\operatorname{dist}(\phi(\overline{n}),\phi(\overline{m})) &\leq \operatorname{dist}(\phi(\overline{n}),x) + \operatorname{dist}(x,\phi(\overline{m}))\\ &= \operatorname{level}(\phi(\overline{n})) - \operatorname{level}(x) + \operatorname{level}(\phi(\overline{m})) - \operatorname{level}(x)\\ &\leq \operatorname{level}(\phi(\overline{n})) - \operatorname{level}(\phi(\overline{o})) + \operatorname{level}(\phi(\overline{m})) - \operatorname{level}(\phi(\overline{o}))\\ &= \operatorname{level}(\overline{n}) - \operatorname{level}(\overline{o}) + \operatorname{level}(\overline{m}) - \operatorname{level}(\overline{o})\\ &= \operatorname{dist}_{\mathbb{T}_\infty}(\overline{n},\overline{m}).\end{aligned}$$

(3) Let $x < y \in \mathbb{M}_\infty$, and let $\overline{n} \in \phi^{-1}(x)$. Choose a path in $\mathbb{M}_\infty$ from x *down* to y. Record (every time such is defined) all the fraternity orders of all elements along that path. Then, starting from x, lift that path node by node, inductively on the level of the node, as follows:
  - $\phi(\overline{n}) = x$.
  - Assume all nodes on the path at distance less than or equal to k from x have been lifted. Let z be on the path, at distance $k+1$ from x. Let z' be the immediate ancestor of z (which is well-defined since we are following a given path), and let $\overline{n'} \in \phi^{-1}(z')$ be the lift of z' chosen according to the induction hypothesis. We consider two cases.
    * If z' is a non-branching node, then lift z to any immediate descendant of $\overline{n'}$.
    * If z' is a branching node, then lift z to the immediate descendant of $\overline{n'}$ of the same fraternal order in $\mathbb{T}_\infty$ as z is among all immediate descendants of z'.
  - Doing this way, we eventually get to an element $\overline{m}$ of $\mathbb{M}_\infty$ such that $\phi(\overline{m}) = y$ with $\overline{m} > \overline{n}$ and $\operatorname{dist}(x,y) = \operatorname{dist}(\overline{n},\overline{m})$.

□

2.3. **Fork argument.**

The fork argument was introduced in [24, proof of Theorem 4.1] for maps that are coarse co-Lipschitz. Here we reduce its hypothesis when we apply it to a map λ that is only ancestor-to-descendant coarse co-Lipschitz.

First let us fix some notation. Consider a metric space $\mathcal{S}$ and a surjective map $\lambda : \mathcal{S} \longrightarrow \mathbb{M}_\infty$ that is ancestor-to-descendant coarse co-Lipschitz. For $d > 0$, consider the (nonempty) set $\mathcal{C}(d)$ of all $c > 0$ such that for all $R \geq d$, for all $\sigma \in \mathcal{S}$, and for all $y > \lambda(\sigma)$ with $\mathrm{dist}(\lambda(\sigma), y) < cR$, there exists $\sigma' \in \mathcal{S}$ with $\|\sigma - \sigma'\| \leq R$ and $\lambda(\sigma') = y$. Denote by c_d^{ATD} the supremum of the set $\mathcal{C}(d)$. It is left to the reader to check (see also [24, Lemma 3.3]) that c_d^{ATD} is attained, and that the family $\left(c_d^{\mathrm{ATD}}\right)_{d>0}$ is increasing. Denote by $c_\infty^{\mathrm{ATD}} := \sup\left\{c_d^{\mathrm{ATD}} : d > 0\right\} = \lim_{d\to\infty} c_d^{\mathrm{ATD}}$.

With these notation, we present the fork argument for ancestor-to-descendant coarse co-Lipschitz maps.

Proposition 2.3.1. *For a metric space $\mathcal{S}$, let $\lambda : \mathcal{S} \longrightarrow \mathbb{M}_\infty$ be a surjective map that is ancestor-to-descendant co-Lipschitz for large distances. Assume that the quantity c_∞^{ATD} for the map λ is finite. Then for every $\eta > 0$, one can find $r > 0$ as large as we want, as well as elements x_0, x_1, and $(x_{2,k})_{k\geq 1}$ in $\mathbb{M}_\infty$, and elements σ_0, σ_1, and $(\sigma_{2,k})_{k\geq 1}$ in $\mathcal{S}$ so that $\lambda(\sigma_0) = x_0$, $\lambda(\sigma_1) = x_1$, and $\lambda(\sigma_{2,k}) = x_{2,k}$ for every k; with the σ's and the x's sitting in an (approximate) fork position as follows:*

$$\mathrm{dist}(x_0, x_1) = \mathrm{dist}(x_1, x_{2,k}) = \frac{\mathrm{dist}(x_0, x_{2,k})}{2} = r;$$

and

$$\begin{aligned}\mathrm{dist}(\sigma_0, \sigma_1) &\leq (1+\eta)\,\frac{r}{c_\infty^{\mathrm{ATD}}},\\ \mathrm{dist}(\sigma_1, \sigma_{2,k}) &\leq (1+\eta)\,\frac{r}{c_\infty^{\mathrm{ATD}}},\\ \frac{\mathrm{dist}(\sigma_0, \sigma_{2,k})}{2} &> (1-\eta)\,\frac{r}{c_\infty^{\mathrm{ATD}}}.\end{aligned}$$

Proof. Let $\varepsilon > 0$. Let $d_0 = \dfrac{3^n}{c_\infty^{\mathrm{ATD}}}$ be large enough so that $c_{d_0}^{\mathrm{ATD}} \leq c_\infty^{\mathrm{ATD}} < (1+\varepsilon)c_{d_0}^{\mathrm{ATD}}$. Then, for any $d \geq d_0$ (for example for $d \geq 81d_0$), we have

$$c_{d_0}^{\mathrm{ATD}} \leq c_d^{\mathrm{ATD}} \leq c_\infty^{\mathrm{ATD}} < (1+\varepsilon)c_{d_0}^{\mathrm{ATD}} \leq (1+\varepsilon)c_d^{\mathrm{ATD}} \leq (1+\varepsilon)c_\infty^{\mathrm{ATD}}.$$

Since c_d^{ATD} is a maximum, and since $(1+\varepsilon)c_d^{\mathrm{ATD}} > c_d^{\mathrm{ATD}}$, we can find a number $R \geq d$, and elements $\sigma \in \mathcal{S}$ and $y > \lambda(\sigma) \in \mathbb{M}_\infty$ with $\mathrm{dist}(y, \lambda(\sigma)) < (1+\varepsilon)c_d^{\mathrm{ATD}}R$ such that

$$\forall \sigma' \in \mathcal{S},\ (\lambda(\sigma') = y) \implies (\|\sigma - \sigma'\| > R)\,.$$

On the other hand, the ancestor-to-descendant coarse co-Lipschitz condition on λ necessitates that $\mathrm{dist}(y, \lambda(\sigma)) \geq c_d^{\mathrm{ATD}}R$. In passing, note that we can then make our of choice of d_0 in such a way that $\mathrm{dist}(y, \lambda(\sigma))$ is as large as we want.

Let us call $x := \lambda(\sigma)$. Let $N \in \mathbb{N}$ be such that $3^N \leq \dfrac{d(x,y)}{2} < 3^{N+1}$. Since y is a descendant of x, we have $\mathrm{dist}(x, y) = \mathrm{level}(y) - \mathrm{level}(x)$, and hence there must exist an integer $p \geq 0$ such that $\mathrm{level}(x) \leq p3^N < (p+1)3^N \leq \mathrm{level}(y)$. In fact, one can take p to be the largest integer such that $(p-1)3^N < \mathrm{level}(x)$. Furthermore, since $d \geq 81d_0$, we must necessarily have $3^n < 3^{N-2}$. In fact, $d \geq 54(1+\varepsilon)d_0$ if ε is small enough. Then since $(1+\varepsilon) \geq c_\infty^{\mathrm{ATD}}/c_d^{\mathrm{ATD}}$, and since $c_\infty^{\mathrm{ATD}}d_0 = 3^n$, we have

$$2 \cdot 3^{N+1} > \mathrm{dist}(x, y) \geq c_d^{\mathrm{ATD}}R \geq c_d^{\mathrm{ATD}}d \geq 6 \cdot 9 \cdot 3^n.$$

We will now identify the elements x_0, x_1 and $x_{2,k}(k \geq 1)$ of $\mathbb{M}_\infty$ that we need for our purpose. First, we fix a path $\mathcal{P}$ downward from x to y. We consider the elements a and b of $\mathcal{P}$ at levels $p \cdot 3^N$ and $(p+1)3^N$ respectively. Note that the segment $[a,b]$ has length 3^N. Divide $[a,b]$ into three subsegments, and denote by x_0 the element of $\mathcal{P}$ at level $p \cdot 3^N + 3^{N-1}$, and by x_3 the one at level $p \cdot 3^N + 2 \cdot 3^{N-1}$. Note that $\operatorname{dist}(x, x_0) \geq 3^{N-1}$ and $\operatorname{dist}(x_3, y) \geq 3^{N-1}$. Divide the segment $[x_0, x_3]$ into three again, and consider the following vertices: the element x_1 of $\mathcal{P}$ at level $p \cdot 3^N + 3^{N-1} + 3^{N-2}$, and the elements $x_{2,k}$ $(k \geq 1)$ of $\mathbb{M}_\infty$ at level $p \cdot 3^N + 3^{N-1} + 2 \cdot 3^{N-2}$ that lie between x_1 and the element x_3 already previously mentioned. Note that although only one of the $x_{2,k}$'s belongs to the original path $\mathcal{P}$, the succession $x < x_0 < x_1 < x_{2,k} < x_3 < y$ can be extended to a shortest path from x to y for any $k \geq 1$.

For notation, let us set

$$\begin{cases} \operatorname{dist}(x, x_0) := D, \\ \\ \operatorname{dist}(x_1, x_0) = \operatorname{dist}(x_{2,k}, x_1) = \operatorname{dist}(x_3, x_{2,k}) := r \; (= 3^{N-2} \geq c_\infty^{\mathrm{ATD}} d_0), \\ \\ \operatorname{dist}(x_3, y) = d(x,y) - D - 3r. \end{cases}$$

We will lift these elements one by one starting from x using the ancestor-to-descendant coarse co-Lipschitz condition.

(1) $\operatorname{dist}(x, x_0) = D < c_{d_0}^{\mathrm{ATD}} \cdot \dfrac{(1+\varepsilon)}{c_{d_0}^{\mathrm{ATD}}} D$; and $\dfrac{(1+\varepsilon)}{c_{d_0}^{\mathrm{ATD}}} D \geq d_0$ because $D \geq 3^{N-1} \geq c_\infty^{\mathrm{ATD}} d_0 \geq c_{d_0}^{\mathrm{ATD}} d_0$. So there exists an element $\sigma_0 \in \mathcal{S}$ with $\operatorname{dist}(\sigma, \sigma_0) \leq \dfrac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} D$ such that $\lambda(\sigma_0) = x_0$.

(2) $\operatorname{dist}(x_0, x_1) = r < c_{d_0}^{\mathrm{ATD}} \cdot \dfrac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} r$; and $\dfrac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} r \geq d_0$ since $r = 3^{N-2} \geq c_\infty^{\mathrm{ATD}} d_0$. So there is an element $\sigma_1 \in \mathcal{S}$ with $\operatorname{dist}(\sigma_0, \sigma_1) \leq \dfrac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} r$ such that $\lambda(\sigma_1) = x_1$.

(3) Similarly, $\operatorname{dist}(x_1, x_{2,k}) = r$ for each $k \geq 1$, so there is an element $\sigma_{2,k} \in \mathcal{S}$ with $\operatorname{dist}(\sigma_1, \sigma_{2,k}) \leq \dfrac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} r$ such that $\lambda(\sigma_{2,k}) = x_{2,k}$.

(4) Finally, $\operatorname{dist}(x_{2,k}, y) = \operatorname{dist}(x,y) - D - 2r < c_{d_0}^{\mathrm{ATD}} \cdot \dfrac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} [\operatorname{dist}(x,y) - D - 2r]$; and $\dfrac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} [\operatorname{dist}(x,y) - D - 2r] \geq \dfrac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} r \geq d_0$, so there exists an element $\sigma^{(k)} \in \mathcal{S}$ with $\operatorname{dist}(\sigma_{2,k}, \sigma^{(k)}) \leq \dfrac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} [\operatorname{dist}(x,y) - D - 2r]$ such that $\lambda(\sigma^{(k)}) = y$.

Now, by the choice of x and y, we have $\operatorname{dist}(\sigma,\sigma^{(k)}) > R$. So

$$
\begin{aligned}
R < \operatorname{dist}(\sigma,\sigma^{(k)}) &\le \frac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}}\left[D + r + r + (\operatorname{dist}(x,y) - D - 2r)\right] \\
&= \frac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}}\operatorname{dist}(x,y) \\
&< \frac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}}(1+\varepsilon)c_{\infty}^{\mathrm{ATD}} R \\
&< (1+\varepsilon)^3 R.
\end{aligned}
$$

As a result, we must have

$$
\operatorname{dist}(\sigma_{2,k},\sigma_0) > [1 - (3^4 - 1)\varepsilon]\cdot\frac{2r}{c_{d_0}^{\mathrm{ATD}}}.
$$

In fact, if $\operatorname{dist}(\sigma_{2,k},\sigma_0) \le \alpha\cdot\frac{2r}{c_{d_0}^{\mathrm{ATD}}}$, then

$$
\begin{aligned}
R &< \operatorname{dist}(\sigma,\sigma^{(k)}) \\
&\le \operatorname{dist}(\sigma,\sigma_0) + \operatorname{dist}(\sigma_0,\sigma_{2,k}) + \operatorname{dist}(\sigma_{2,k},\sigma^{(k)}) \\
&\le \frac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}} D + \frac{\alpha}{c_{d_0}^{\mathrm{ATD}}} 2r + \frac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}}(\operatorname{dist}(x,y) - D - 2r) \\
&= \frac{\alpha}{c_{d_0}^{\mathrm{ATD}}} 2r + \frac{1+\varepsilon}{c_{d_0}^{\mathrm{ATD}}}(\operatorname{dist}(x,y) - 2r) \\
&= \frac{\operatorname{dist}(x,y)}{c_{d_0}^{\mathrm{ATD}}}\left[t\alpha + (1-t)(1+\varepsilon)\right], \text{ where } t = \frac{2r}{\operatorname{dist}(x,y)} \\
&< \frac{(1+\varepsilon)c_{\infty}^{\mathrm{ATD}} R}{c_{d_0}^{\mathrm{ATD}}}\left[t\alpha + (1-t)(1+\varepsilon)\right] \\
&< (1+\varepsilon)^2[t\alpha + (1-t)(1+\varepsilon)]R.
\end{aligned}
$$

This would require that $\alpha > (1+\varepsilon) - \frac{1}{t}\left[(1+\varepsilon) - \frac{1}{(1+\varepsilon)^2}\right]$. But from the definition of N, we have $\operatorname{dist}(x,y) < 2\cdot 3^{N+1}$. So, since $r = 3^{N-2}$, we have

$$
\frac{1}{t} = \frac{\operatorname{dist}(x,y)}{2r} < 3^3,
$$

and hence

$$\alpha > (1+\varepsilon) - 3^3\left[(1+\varepsilon) - \frac{1}{(1+\varepsilon)^2}\right] \geq (1+\varepsilon) - 3^3(3\varepsilon).$$

To summarize, we have $\sigma_0 \in \lambda^{-1}(x_0)$, $\sigma_1 \in \lambda^{-1}(x_1)$, and $\sigma_{2,k} \in \lambda^{-1}(x_{2,k})$ $(k \geq 1)$, with

$$\operatorname{dist}(\sigma_0, \sigma_1) \leq (1+\varepsilon)\,\frac{r}{c_{d_0}^{\mathrm{ATD}}},$$

and

$$\operatorname{dist}(\sigma_1, \sigma_{2,k}) \leq (1+\varepsilon)\,\frac{r}{c_{d_0}^{\mathrm{ATD}}},$$

but

$$\operatorname{dist}(\sigma_0, \sigma_{2,k}) > (1-80\varepsilon)\,\frac{2r}{c_{d_0}^{\mathrm{ATD}}}.$$

This finishes the proof. □

2.4. **Proof of Proposition 2.0.2.**

We apply Proposition 2.3.1 to the set $\mathcal{S} := f^{-1}(\Sigma) = f^{-1}\left(j^{-1}(\mathbb{T}_\infty)\right)$ and the map $\lambda = \phi \circ j \circ f_{|\mathcal{S}}$. Recall that this particular map is also Lipschitz (for large distances), and hence the hypothesis of Proposition 2.3.1 applies since $c_d^{\mathrm{ATD}} \leq \operatorname{Lip}(\lambda)$ for all $d > 0$. We find the elements $x_0, x_1, (x_{2,k})_{k\geq 1} \in \mathbb{M}_\infty$, and $\sigma_0, \sigma_1, (\sigma_{2,k})_{k\geq 1} \in \mathcal{S} \subseteq X$ as given by Proposition 2.3.1. We have for all $k \geq 1$,

$$\|\sigma_0 - \sigma_1\| \leq (1+\varepsilon)\,\frac{r}{c_{d_0}^{\mathrm{ATD}}},$$

and

$$\|\sigma_1 - \sigma_{2,k}\| \leq (1+\varepsilon)\,\frac{r}{c_{d_0}^{\mathrm{ATD}}},$$

but

$$\|\sigma_0 - \sigma_{2,k}\| > (1-80\varepsilon)\,\frac{2r}{c_{d_0}^{\mathrm{ATD}}}.$$

On the other hand, since λ is Lipschitz , we have for all $k \neq l$,

$$\|\sigma_{2,k} - \sigma_{2,l}\| \geq \frac{1}{\operatorname{Lip}(\lambda)}\operatorname{dist}(x_{2,k}, x_{2,l}) = \frac{1}{\operatorname{Lip}(\lambda)}\,2r.$$

Since X has property (β), this implies that we must have

$$\overline{\beta}_X\left(\frac{c_\infty^{\mathrm{ATD}}}{\operatorname{Lip}(\lambda)}\right) \leq \overline{\beta}_X\left(\frac{2c_{d_0}^{\mathrm{ATD}}}{(1+\varepsilon)\operatorname{Lip}(\lambda)}\right) \leq 81\varepsilon.$$

Since ε is arbitrary, we then get that

$$\overline{\beta}_X\left(\frac{c_\infty^{\mathrm{ATD}}}{\operatorname{Lip}(\lambda)}\right) = 0.$$

This contradiction finishes the proof.

□

3. Removing the reflexivity assumption

James proves many characterizations of reflexivity in [13]. We highlight in the following lemma the one that we need. For this purpose, denote by $\mathbb{T}$ the set of all finite subsets of $\mathbb{N}$, considered in graph-theoretic terms as a tree. Note that this time $\mathbb{T}$ is only considered as a tree to organize its elements, but we do not put any metric on $\mathbb{T}$. As a set, we have $\mathbb{T} = \mathbb{T}_\infty$.

Lemma 3.0.1 (James)**.** *Let X be a non-reflexive Banach space. Then for every $0 < \theta < 1$, there exists a sequence $(u_n)_{n\geq 1} \subseteq B_X$ such that*

(1) *the map from $\mathbb{T}$ to X given by $\{n_1 < n_2 < \cdots < n_k\} \longmapsto u_{n_1} + \cdots + u_{n_k}$ is one-to-one,*
(2) *for every $n_1 < \cdots < n_k$, we have*
$$\theta k \leq \|u_{n_1} + \cdots + u_{n_k}\| \leq k,$$
(3) *and for every $n_1 < \cdots < n_k < m_1 < \cdots < m_l$, we have*
$$\frac{\theta}{3}(k+l) \leq \|(u_{n_1} + \cdots + u_{n_k}) - (u_{m_1} + \cdots + u_{m_l})\| \leq k+l.$$

Proof. Let $\theta \in (0,1)$. From item (31) in [13], we can find sequences $(u_n)_{n\geq 1} \subseteq B_X$ and $(u_n^*)_{n\geq 1} \subseteq B_{X^*}$ such that $u_n^*(u_k) = \theta$ if $n \leq k$, and $u_n^*(u_k) = 0$ if $n > k$.

(1) Let $\{n_1 < \cdots < n_k\} \neq \{m_1 < \cdots < m_l\}$. Denote $\{n_1' < \cdots < n_{k+l}'\} = \{n_1 < \cdots < n_k\} \cup \{m_1 < \cdots < m_l\}$, and let signs $(\varepsilon_i)_{1\leq i\leq k+l} \subseteq \{-1,1\}$ be such that
$$(u_{n_1} + \cdots + u_{n_k}) - (u_{m_1} + \cdots + u_{m_l}) = \varepsilon_1 u_{n_1'} + \cdots + \varepsilon_{k+l} u_{n_{k+l}'}.$$
Then we have
$$\|(u_{n_1}+\cdots+u_{n_k})-(u_{m_1}+\cdots+u_{m_l})\| \geq \left|u_{n_{k+l}'}\left(\varepsilon_1 u_{n_1'} + \cdots + \varepsilon_{k+l} u_{n_{k+l}'}\right)\right| = |\varepsilon_{k+l}\,\theta| > 0.$$

(2) For $n_1 < \cdots < n_k$, we have:
$$\|u_{n_1} + \cdots + u_{n_k}\| \geq u_{n_1}^*(u_{n_1} + \cdots + u_{n_k}) = \theta\, k.$$
The other inequality follows from the triangle inequality.

(3) Let $n_1 < \cdots < n_k < m_1 < \cdots < m_l$. On the one hand, we have
$$\|(u_{n_1}+\cdots+u_{n_k})-(u_{m_1}+\cdots+u_{m_l})\| \geq \left|u_{m_1}^*\left((u_{n_1} + \cdots + u_{n_k}) - (u_{m_1} + \cdots + u_{m_l})\right)\right| = \theta\, l,$$
and on the other hand,
$$\|(u_{n_1}+\cdots+u_{n_k})-(u_{m_1}+\cdots+u_{m_l})\| \geq \left|u_{n_1}^*\left((u_{n_1} + \cdots + u_{n_k}) - (u_{m_1} + \cdots + u_{m_l})\right)\right| = \theta|k-l|.$$
We get the left inequality by noting that $\max\{l, |k-l|\} \geq \frac{1}{3}(k+l)$. Again the inequality on the right follows from the triangle inequality.

□

We now state the main result in this section.

Theorem 3.0.2. *Assume that a Banach space Y is a uniform quotient of a Banach space X. If X has (an equivalent norm with) property (β), then Y has to be reflexive.*

Proof. Let $f : X \longrightarrow Y$ be a surjective uniform quotient map. Note that f is coarse Lipschitz and coarse co-Lipschitz. Assume that Y is not reflexive. We apply James' characterization with $\theta = 3/4$. For $J = \{n_1 < n_2 < \ldots < n_k\} \in \mathbb{T}$, set $v_J := u_{n_1} + \ldots + u_{n_k}$. We then get that if $\max J < \min J'$, then

$$\begin{cases} \dfrac{1}{4}|J| \leq \|v_J\| \leq |J|, \\ \\ \dfrac{1}{4}|J'| \leq \|v_{J'}\| \leq |J'|, \\ \\ \dfrac{1}{4}\left(|J| + |J'|\right) \leq \|v_J - v_{J'}\| \leq \left(|J| + |J'|\right). \end{cases} \tag{1}$$

Consider the subset $\mathcal{T} := \{v_J \in Y : J \in \mathbb{T}\} \subseteq Y$, which *is* a metric space. Since the correspondence $J \longmapsto v_J$ is one-to-one, the set $\mathcal{T}$ is in one-to-one and onto correspondence with $\mathbb{T}$, and we extend the notions of levels, ancestors, and descendants to elements of $\mathcal{T}$ as well. Consider the composition

$$\widetilde{\phi} := (\phi \circ \equiv) : \mathcal{T} \ \equiv \ \mathbb{T}_\infty \longrightarrow \mathbb{M}_\infty,$$

where ϕ is the same map as in subsection 2.2, and $\equiv$ is the natural bijection between $\mathcal{T}$ and $\mathbb{T}_\infty$. The bijection $\mathcal{T} \equiv \mathbb{T}_\infty$ is ancestor-to-descendant bi-Lipschitz by James' characterization as expressed in (1) above. And hence, the composition $\widetilde{\phi} : \mathcal{T} \longrightarrow \mathbb{M}_\infty$, considered as a map between metric spaces, is ancestor-to-descendant co-Lipschitz.

Next, consider the restriction $f_{|_{f^{-1}(\mathcal{T})}} : f^{-1}(\mathcal{T}) \longrightarrow \mathcal{T} \subseteq Y$, which is a surjective uniform quotient mapping that is coarse Lipschitz and coarse co-Lipschitz. As a result, the composition $\widetilde{\lambda} := \widetilde{\phi} \circ f_{|_{f^{-1}(\mathcal{T})}} : f^{-1}(\mathcal{T}) \longrightarrow \mathbb{M}_\infty$ is also ancestor-to-descendant co-Lipschitz. Note that $c_\infty^{\mathrm{ATD}}(\widetilde{\lambda})$ is finite since the inequalities (1) also give that the map $\widetilde{\phi}$ is ancestor-to-descendant Lipschitz. We can thus apply the fork argument (Proposition 2.3.1) to $\widetilde{\lambda}$, producing elements x_0, x_1, $x_{2,k}$ $(k \geq 1)$, x_3 in $\mathbb{M}_\infty$ with $\mathrm{dist}(x_0, x_1) = \mathrm{dist}(x_1, x_{2,k}) = \mathrm{dist}(x_{2,k}, x_3) = r = 3^{N-2}$; as well as respective $\widetilde{\lambda}$-preimages σ_0, σ_1, $\sigma_{2,k}$ $(k \geq 1)$ in $f^{-1}(\mathcal{T}) \subseteq X$ with

$$\begin{cases} \|\sigma_0 - \sigma_1\|, \|\sigma_1 - \sigma_{2,k}\| \leq (1+\varepsilon)\, \dfrac{r}{c_{d_0}^{\mathrm{ATD}}(\widetilde{\lambda})} \\ \\ \|\sigma_0 - \sigma_{2,k}\| > (1 - 80\varepsilon)\, \dfrac{2r}{c_{d_0}^{\mathrm{ATD}}(\widetilde{\lambda})}. \end{cases} \tag{2}$$

Let us denote $v^{(0)} := f(\sigma_0)$, $v^{(1)} := f(\sigma_1)$, and $v_{J^{(k)}} := f(\sigma_{2,k})$ (for all $k \geq 1$), which are elements of the subset $\mathcal{T}$ of Y. Recall that the map ϕ preserves levels, so

$$\begin{cases} \mathrm{level}(v^{(0)}) = \mathrm{level}(x_0), \\ \\ \mathrm{level}(v^{(1)}) = \mathrm{level}(x_1) = \mathrm{level}(x_0) + 3^{N-2}, \\ \\ \mathrm{level}(v_{J^{(k)}}) = \mathrm{level}(x_{2,k}) = \mathrm{level}(x_1) + 3^{N-2}. \end{cases}$$

But $\mathrm{level}(v_{J^{(k)}}) = |J^{(k)}|$, so $|J^{(k)}| = \mathrm{level}(x_1) + 3^{N-2}$ for all $k \geq 1$.

Consider the set of all elements of $\mathbb{T}$ with infinitely (not necessarily immediate) descendants among the $J^{(k)}$'s. This set is nonempty since the root $\emptyset$ belongs to it. Now let J be an element of this set such that level$(J) = |J|$ is maximal. Let $\left(J^{(k_n)}\right)_{n\geq 1}$ be the infinitely many descendants of J among the $J^{(k)}$'s. Write $J^{(k_n)} =: J \cup I^{(n)}$ with $\max J < \min I^{(n)}$. Then for every $n \geq 1$, we must have

$$|I^{(n)}| \geq 3^{N-2}. \tag{3}$$

In fact, the map $\phi : \mathbb{T}_\infty \longrightarrow \mathbb{M}_\infty$ is 1-Lipschitz, so for any $k \neq j$ we have

$$2 \cdot 3^{N-2} = \mathrm{dist}_{\mathbb{M}_\infty}(x_{2,k}, x_{2,j}) \leq \mathrm{dist}_{\mathbb{T}_\infty}(J^{(k)}, J^{(j)}),$$

i.e. the closest common ancestor to $J^{(k)}$ and $J^{(j)}$ is at least 3^{N-2} generations back.

Next, consider the immediate descendants $(J \cup \{m\})_{m>\max(J)}$ of J. By the fact that level(J) is maximal among the elements of $\mathbb{T}$ with infinitely many descendants among the $J^{(k)}$'s, we must find an infinite sequence $m_1 < m_2 < \cdots$ such that

- each $J \cup \{m_j\}$ has descendants among the $J^{(k_n)}$'s,
- and each such $J \cup \{m_j\}$ has only finitely many descendants among the $J^{(k_n)}$'s.

Take a further subsequence of $(m_j)_{j\geq 1}$, still denoted $(m_j)_{j\geq 1}$, such that if we denote by $J^{\left(k_{n_j}\right)} = J \cup I^{(n_j)}$ the descendant of $J \cup \{m_j\}$ with $\max\left(I^{(n_j)}\right)$ being maximal, then $\max(I^{(n_j)}) < m_{j+1} = \min(I^{(n_{j+1})})$.

Then, by James, we have

$$\left\| v_{J^{(k_{n_j})}} - v_{J^{(k_{n_{j'}})}} \right\| \geq \frac{1}{4}\left(|I^{(n_j)}| + |I^{(n_{j'})}|\right) \geq \frac{1}{2} \cdot 3^{N-2}.$$

(The last inequality comes from inequality (3).) Using the Lipschitz condition for large distances on f, we get

$$\left\| \sigma_{2,k_{n_j}} - \sigma_{2,k_{n_{j'}}} \right\| \geq \frac{1}{L} \cdot 3^{N-2} = \frac{1}{L} \cdot r$$

for some constant $L > 0$ (for example take $L/2$ to be the Lipschitz constant for f for distances larger than $\inf\{\|\sigma_{2,k_{n_j}} - \sigma_{2,k_{n_{j'}}}\| : j \neq j'\}$, which is strictly positive since f is uniformly continuous).

Combining this with the distances expressed in (2), and using property (β) on X, we must have

$$\overline{\beta}_X\left(\frac{c_\infty^{\mathrm{ATD}}(\widetilde{\lambda})}{2L}\right) \leq \overline{\beta}_X\left(\frac{c_{d_0}^{\mathrm{ATD}}(\widetilde{\lambda})}{(1+\varepsilon)L}\right) \leq 81\varepsilon.$$

Since ε is arbitrary, we then get that

$$\overline{\beta}_X\left(\frac{c_\infty^{\mathrm{ATD}}(\widetilde{\lambda})}{2L}\right) = 0,$$

a contradiction. □

Remark 3.0.3. A superreflexive Banach space is one that has an equivalent norm which is uniformly convex (see for example [5, Appendix A]). A result of Bates, Johnson, Lindenstrauss, Preiss, and Schechtman [2] shows that a Banach space that is a uniform quotient of a superreflexive Banach space is itself superreflexive. Our result (by combining Theorem 2.0.1 and Theorem 3.0.2) gives a generalization of this for the case of separable Banach spaces. In fact, a uniformly convex Banach space has property (β) (see [30]); but there are separable spaces with property (β) which are not superreflexive (see for example [19]).

In [4] Baudier, Kalton and Lancien show that a Banach space which coarse Lipschitz embeds into a separable Banach space with an equivalent norm with property (β) also has an equivalent norm with property (β). Our result here gives the analog of this for uniform quotients.

4. Quantitative results

Recall the following asymptotic moduli.

Definition 4.0.4. Let X be an infinite-dimensional Banach space. Let $t \in (0,1]$.

(1) Modulus of asymptotic uniform convexity (AUC) [12,15,26]:

$$\overline{\delta}(t) = \inf_{\|x\|=1} \sup_{X_0} \inf_{\substack{z\in X_0\\ \|z\|\geq t}} \{\|x+z\|-1\},$$

where X_0 runs through all closed subspaces of X of finite codimension. The Banach space X is said to be asymptotically uniformly convex (AUC) if $\overline{\delta}(t) > 0$ for $t > 0$. A reflexive Banach space that is AUC is in particular called nearly uniformly convex (NUC).

(2) Modulus of asymptotic uniform smoothness (AUS) [15,26,29]:

$$\overline{\rho}(t) = \sup_{\|x\|=1} \inf_{X_0} \sup_{\substack{z\in X_0\\ \|z\|\leq t}} \{\|x+z\|-1\},$$

where X_0 runs through all closed subspaces of X of finite codimension. The Banach space X is said to be asymptotically uniformly smooth(AUS) if $\lim\limits_{t\to 0} \overline{\rho}(t)/t = 0$. A reflexive Banach space that is AUS is in particular called nearly uniformly smooth (NUS).

(3) (β)-modulus [1]:

$$\overline{\beta}(t) = 1 - \sup\left\{\inf\left\{\frac{\|x+x_n\|}{2} : n\in\mathbb{N}\right\}\right\},$$

where the supremum is taken over all $x \in B_X$ and all sequences $(x_n)_{n\geq 1} \subseteq B_X$ with $\operatorname{sep}((x_n)_{n\geq 1}) \geq t$. The Banach space X is said to have property (β) if $\overline{\beta}(t) > 0$ for $t > 0$.

Lemma 4.0.5. *For any infinite-dimensional Banach space X, and any $t \in (0,1/2]$, we have $\overline{\beta}_X(t) \leq \overline{\delta}_X(2t)$.*

Proof. For $t \in (0,1/2]$, denote $\beta = \overline{\beta}_X(t) > 0$. Let $x \in S_X$. Then for every sequence $(u_n)_{n\geq 1} \subseteq B_X$ with $\operatorname{sep}((u_n)_{n\geq 1}) \geq t$, one has

$$\inf\left\{\left\|\frac{x+u_n}{2}\right\| : n\geq 1\right\} \leq 1-\beta < \frac{1}{1+\beta}.$$

Multiplying everything by $1+\beta$ and taking the contrapositive, we get for any sequence $(x_n)_{n\geq 1}$ that if $\operatorname{sep}((x_n)_{n\geq 1}) \geq (1+\beta)\,t$, and if $\inf\limits_{n\geq 1} \|(1+\beta)x + x_n\| \geq 2$, then $\|x_n\| > 1+\beta$ on a subsequence, and hence $\liminf \|x_n\| \geq 1+\beta$.

Denote $T = 2t$. For a finite codimensional subspace X_0, denote

$$\overline{\delta}_X(T, x, X_0) = \inf\{\|x+z\|-1 : z\in X_0, \|z\|\geq T\};$$

and denote $\overline{\delta}_X(T,x)$ the supremum of $\overline{\delta}_X(T,x,X_0)$ over all finite codimensional subspaces X_0.

Fix $\varepsilon > 0$. Let $x^* \in S_{X^*}$ be such that $x^*(x) = \|x\| = 1$. Denote by $X_1 = \ker(x^*)$. Let $z_1 \in X_1$ be such that $\|z_1\| \geq T$ and

$$\|x + z_1\| - 1 < \overline{\delta}_X(T,x,X_1) + \varepsilon \leq \overline{\delta}_X(T,x) + \varepsilon.$$

Let $z_1^* \in S_{X^*}$ be such that $z_1^*(z_1) = \|z_1\|$. Call $X_2 = X_1 \cap \ker(z_1^*)$, and let $z_2 \in X_2$ be such that $\|z_2\| \geq T$ and

$$\|x + z_2\| - 1 < \overline{\delta}_X(T,x,X_2) + \varepsilon \leq \overline{\delta}_X(T,x) + \varepsilon.$$

Continuing by induction, we find a sequence $(z_n)_{n\geq 1} \subseteq \ker(x^*)$ and functionals $(z_n^*)_{n\geq 1} \subseteq S_{X^*}$ such that for every $n \geq 1$ one has $z_n^*(z_n) = \|z_n\| \geq T$, and $z_{n+1} \in \bigcap_{i=1}^n \ker(z_i^*)$, and

$$\|x + z_n\| - 1 \leq \overline{\delta}_X(T,x) + \varepsilon.$$

Denote $x_n = x + z_n$. Then for $m > n$,

$$\|x_n - x_m\| = \|z_n - z_m\| \geq z_n^*(z_n - z_m) = z_n^*(z_n) = \|z_n\| \geq T.$$

So $\operatorname{sep}((x_n)_{n\geq 1}) \geq T = 2t \geq (1+\beta)\,t$. (Note that $\beta \leq 1$ by definition.) On the other hand, we have

$$\|(1+\beta)x + x_n\| = \|(2+\beta)x + z_n\| \geq x^*((2+\beta)x + z_n) = 2 + \beta \geq 2.$$

As a result, we must have $\liminf \|x_n\| = \liminf \|x + z_n\| \geq 1 + \beta$.

This implies that $\beta = \overline{\beta}_X(t) \leq \overline{\delta}_X(2t,x) + \varepsilon$, and since $\varepsilon > 0$ and $x \in S_X$ are arbitrary, we get $\overline{\beta}_X(t) \leq \overline{\delta}_X(2t)$.

□

Definition 4.0.6. We say that the (β)-modulus has power type p if there is a constant $C > 0$ so that $\overline{\beta}(t) \geq Ct^p$.

Proposition 4.0.7. *Let $1 < p < \infty$. There exist uniformly homeomorphic Banach spaces Z and Y so that Z has (β)-modulus of power type p while Y has no equivalent norm with (β)-modulus of power type p.*

Proof. Let $(G_n)_{n\geq 1}$ be a sequence of finite-dimensional Banach spaces dense in all finite-dimensional Banach spaces for the Banach-Mazur distance. Let $1 < p < \infty$. Johnson [14] introduced the space C_p given by $C_p = (\sum_{n=1}^\infty G_n)_{\ell_p}$. Kalton [17] considered the space $X = \left((\sum_{n=1}^\infty G_n)_{T_p}\right)^2$, where T_p is the p-convexification of Tsirelson space T (see for example [7]). Kalton proved in [17] that C_p and X are uniformly homeomorphic.

The computation in [9, Proposition 5.1] gives that C_p has (β)-modulus of power type p. On the other hand, Kalton remarks in [17] that X does not admit any equivalent norm with AUC modulus of power type p. Assume that X admits an equivalent norm with (β)-modulus of power type p, that is $\overline{\beta}_X(t) \geq Ct^p$. By Lemma 4.0.5, we have $\overline{\beta}_X(t) \leq \overline{\delta}_X(2t)$. Thus, for this norm we obtain $\overline{\delta}_X(t) \geq Ct^p$ which is in contradiction with Kalton's result. □

Remark 4.0.8. In contrast with Theorems 2.0.1 and 3.0.2, as well as with [25, Proposition 4.1], the above proposition shows that we cannot compare $\overline{\beta}_Z(t)$ and $\overline{\beta}_Y(t)$ even under renorming, despite the fact that one is a uniform quotient of the other.

Furthermore, it was proved in [9] that if a Banach space Y is a uniform quotient of a Banach space Z, then for some constants C_1 and C_2 we have,

$$C_1\overline{\beta}_Z(C_2 t) \leq \overline{\rho}_Y(t).$$

Taking $Z = C_p$ and $Y = X$, it follows from the above proposition that one can not replace the AUS modulus $\overline{\rho}_Y(t)$ with the AUC modulus $\overline{\delta}_Y(t)$ even under an equivalent norm on Y.

The rest of the article is devoted to an investigation of the following question.

> Assume that two Banach spaces are uniformly homeomorphic and that one of them has (β)-modulus of power type p. Given an arbitrary $q > p$, does the other space always have an equivalent norm with (β)-modulus of power type q?

In the case of the conterexample provided by the pair of spaces C_p and X, we can give a positive answer to the above question.

Proposition 4.0.9. *Assume that X is a separable reflexive Banach space which is uniformly homeomorphic to a space with $\overline{\rho}(t) \leq C_1 t^p$, $1 < p < \infty$, and that the dual space X^* is uniformly homeomorphic to a space with $\overline{\rho}(t) \leq C_2 t^{p'}$ where $1/p + 1/p' = 1$. Then for any $\gamma > 0$, X admits an equivalent norm with $\overline{\beta}(t) \geq Ct^{p+\gamma}$.*

Proof. It is shown in [11] that the space X has an equivalent norm $\|.\|_1$ with $\overline{\rho}_{\|.\|_1}(t)$ of power type $p - \varepsilon$ (see also [18]). Similarly, for any $\varepsilon_1 > 0$, X^* has an equivalent norm $\|.\|_2^*$ with $\overline{\rho}_{\|.\|_2^*}(t)$ of power type $p' - \varepsilon_1$. By duality [29], X has $\overline{\delta}_{\|.\|_2}(t)$ of power type $p+\varepsilon$ provided we choose ε_1 accordingly. Then by [15, Proposition 2.10] (see also [27]), X admits an equivalent norm $\|.\|$ with both $\overline{\rho}(t)$ of power type $p-\varepsilon$ and $\overline{\delta}(t)$ of power type $p + \varepsilon$.

It was shown in [20] that if a norm is both NUC and NUS then it also has property (β). Following the same avenue, it was proved in [10] that a norm with the above power types for $\overline{\rho}(t)$ and $\overline{\delta}(t)$ has a (β)-modulus $\overline{\beta}(t)$ of power type

$$\frac{(p-\varepsilon)(p+\varepsilon) - (p-\varepsilon)}{(p-\varepsilon) - 1} = p + \varepsilon'.$$

Letting ε tend to 0, ε' can be made as close to 0 as we want. $\square$

Corollary 4.0.10. *For every $\varepsilon > 0$, the space $X = \left((\sum_n G_n)_{T_p}\right)^2$ has an equivalent norm with $\overline{\beta}(t)$ of power type $p + \varepsilon$.*

Proof. By [17], X is uniformly homeomorphic to C_p and $\overline{\rho}_{C_p}(t) \leq Ct^p$. On the other hand,

$$X^* = \left((\sum_n G_n^*)_{T_p^*}\right)^2 = \left((\sum_n G_n^*)_{T_{p'}}\right)^2.$$

Since $(G_n^*)_{n\geq 1}$ is also dense in Banach-Mazur metric, we obtain by the same theorem in [17] that $\left((\sum_{n=1}^{\infty} G_n^*)_{T_{p'}}\right)^2$ is uniformly homeomorphic to $C_{p'}$; and $\overline{\rho}_{C_{p'}}(t) \leq Ct^{p'}$. □

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