Exam 4 Review problems

(1) If $f'(x) = (x-1)(x+3)^2(x-5)$, identify the intervals in which the function $f(x)$ is increasing and decreasing. Then, identify the x-coordinates of all of the relative extrema.

Solution: Since $f'(x) = 0$ if $x = -3, 1, 5$ and

$$f'(-4) > 0 \quad f'(0) > 0 \quad f'(2) < 0 \quad f'(6) > 0,$$

$f(x)$ is increasing on the intervals $(-\infty, -3), (-3, 1)$ and $(5, +\infty)$ and decreasing on the interval $(1, 5)$. Also, $f(x)$ has a relative maximum when $x = 1$ and a relative minimum when $x = 5$.

(2) If $f''(x) = x(e^x - 4)(x-1)^2$, identify the intervals in which the function $f(x)$ is concave up and concave down. Then, identify all of the inflection points.

Solution: Since $f''(x) = 0$ if $x = 0, 1, \ln 4$ and

$$f''(-1) > 0 \quad f''(1/2) < 0 \quad f''(\ln 3) < 0 \quad f''(\ln 5) > 0$$

$f(x)$ is concave up on the intervals $(-\infty, 0)$ and $(\ln 4, +\infty)$ and concave down on the intervals $(0, 1)$ and $(1, \ln 4)$. Also, the inflection points are $(0, f(0))$ and $(\ln 4, f(\ln 4))$.

(3) Let $f(x) = \tan(x) - x$. Determine the absolute maximum and absolute minimum value of $f(x)$ for x in the interval $[2\pi/3, 5\pi/4]$(Use a calculator to determine the max and min. You won't need one for the test).

Solution: Since $f'(x) = \sec^2 x - 1$, $f'(x) = 0$ if $\sec^2 x = 1$. This is the same as $1 = \cos^2 x$. Therefore, $x = \pi$ is the only solution in the inteval $[2\pi/3, 5\pi/4]$. Since

$$f(2\pi/3) \approx -3.83 \quad f(\pi) = -\pi \approx -3.14 \quad f(5\pi/4) \approx -2.93$$

the absolute maximum value of $f(x)$ is $f(5\pi/4) \approx -2.93$ and the absolute minimum value of $f(x)$ is $f(2\pi/3) \approx -3.83$.

(4) You are standing at the point $(1, 2)$ and a bee is flying along the curve $y = x - 1$.

(a) Draw a plot with the point $(1, 2)$ and the curve $y = x - 1$ for $x > 0$.

(b) If the bee is at the point (x, y), the distance between you and the bee is

$$D = \sqrt{(x-1)^2 + (y-2)^2}$$

Use part (a) to write a formula for the distance, as a function of x. In other words, write a formula for $D(x)$ if the bee is at the point $(x, x-1)$.

Solution:

$$D(x) = \sqrt{(x-1)^2 + ((x-1)-2)^2} = \sqrt{2x^2 - 8x + 10}$$

(b) Find $D'(x)$.

Solution:

$$D'(x) = (1/2)(2x^2 - 8x + 10)^{-1/2}(4x - 8) = \frac{2x-4}{\sqrt{2x^2-8x+10}}$$

(c) Find the one critical point of $D(x)$ for $x > 0$.

Solution: Since $D'(x) = 0$ when $2x - 4 = 0$, the only critical point occurs when $x = 2$.

(d) What are the coordinates of the location of the bee when it is closest to you? How close does it get to you?

Solution: Since $D(0) = \sqrt{10}$, $D(2) = \sqrt{2}$ and $\lim_{x\to\infty} D(x) = +\infty$, the bee is $\sqrt{2}$ units from you when the bee is located at the point $(2, \sqrt{2})$.

(e) Describe an easier way to determine the minimum distance in part (d) without using calculus?

Solution: The point on the line $y = x - 1$ closest to the point $(1, 2)$ must be on a line containing the point $(1, 2)$ and perpendicular to $y = x - 1$. Therefore, the point is the intersection of the lines $y = x - 1$ and $y - 2 = -(x - 1)$. Therefore, $x - 1 = -x + 3$ or $x = 2$.

(5) Practice setting up the word problems in 5.5. Just determine the function which is to be maximized or minimized and determine the domain of that function.

(6) Let $s(t) = t^3/3 - 5t^2/2 + 4t + 3$ describe the position of a particle in rectilinear motion.

(a) Find the velocity and acceleration functions.

Solution:

$$v(t) = t^2 - 5t + 4 = (t-4)(t-1) \quad a(t) = 2t - 5$$

(b) Identify the times in which the particle is stopped.

Solution: $v(t) = 0$ if $t = 1$ or $t = 4$.

(c) Identify when the particle is speeding up and slowing down.

Solution: $v(t)$ is positive on the intervals $(0, 1)$ and $(4, +\infty)$ and negative on $(1, 4)$. $a(t)$ is positive on $(5/2, +\infty)$ and negative on $(0, 5/2)$. Therefore, the particle is speeding up on $(1, 5/2)$ and $(4, +\infty)$ and slowing down on $(0, 1)$ and $(5/2, 4)$.

(d) Determine the maximum and minimum speed of the particle for $0 \leq t \leq 5$.

Solution: Since $v(t) = 0$ when $t = 1$ or $t = 4$, the minimum speed is $|v(1)| = |v(4)| = 0$. Since $a(t) = 0$ if $t = 5/2$, we have

$$v(0) = 4 \quad v(5/2) = -2.25 \quad v(5) = 4.$$

Therefore, the maximum speed is the maximum of $|4|$ and $|-2.25|$. Therefore, the maximum speed is 4, when $t = 0$ and $t = 5$.