Math 141 Homework Solutions

1. Homework

A.38 The value of S is the area of a circle of radius L minus the area of a circular sector with radius L and angle θ. This comes from the area formula for a circle.

$$S = \pi L^2 - (\theta/2)L^2.$$

The circumference of the base of the cone is equal to the measure of the circumference of a circle with radius L minus the length of the circular arc with radius L and angle θ. This comes from the circumference formula for circles.

$$2\pi r = 2\pi L - \theta L.$$

From the second equation,

$$\theta/2 = (\pi L - \pi r)/L.$$

Therefore, a little algebra yields

$$S = \pi L r.$$

B.14 Some possible zeros or roots of the polynomial $p(x) = 2x^4 + x^3 + 3x^2 + 3x - 9$ are positive or negative $9/2, 9, 3/2, 3, 1/2, 1$. It is easy to verify that

$$p(1) = p(-3/2) = 0.$$

Therefore, $(x - 1)$ and $(x + 3/2)$ are factors of $p(x)$. Since $p(x)$ is a fourth degree polynomial, we know that

$$p(x) = (x - 1)(x + 3/2)(ax^2 + bx + c)$$

for some a, b and c. It should be easy to see that $ax^4 = 2x^4$ and $(-3c)/2 = -9$. Therefore, $a = 2$ and $c = 6$. So,

$$p(x) = (x - 1)(x + 3/2)(2x^2 + bx + 6).$$

Now, we can see that $9x - 6x - 3bx/2 = 3x$ Therefore, $b = 0$ and

$$p(x) = (x - 1)(x + 3/2)(2x^2 + 6).$$

By the quadratic formula, we can see that there are no more factors.

2. Homework

1.1.12(c) Obviously, $x \neq 4$. However, x must also satisy

$$\frac{x^2-4}{x-4} \geq 0.$$

This is the same as

$$\frac{(x-2)(x+2)}{x-4} \geq 0.$$

Therefore, x must satisfy $-2 \leq x \leq 2$ or $x > 4$. So the domain of G is the set of all x such that

$$-2 \leq x \leq 2 \; or \; 4 < x.$$

1.1.20 For (a)

$$y = 4 \rightarrow 3 = \sqrt{x} \rightarrow 9 = x.$$

For (b),

$$y = 0 \rightarrow -1 = \sqrt{x} \rightarrow No\ Solution.$$

For (c),

$$y \geq 6 \rightarrow 5 \leq \sqrt{x} \rightarrow 25 \leq x.$$

For (d), the smallest value of $\sqrt{x}$ is zero. Therefore, the smallest value of y is one.

3. Homework

1.3.2 Since the graph of f is just two connected line segments, you just need to figure out where the points $(-1, 0), (0, 2)$ and $(2, 2)$ are moved to and then connect the dots. For (a), the points become $(1, 0), (0, -2)$ and $(-2, -2)$. For (b), the points become $(3, 0), (2, 2)$ and $(0, 2)$. For (c), the points become $(3, 1), (2, -1)$ and $(0, -1)$. For (d), the points become $(-1/2, 0), (0, 1)$ and $(1, 1)$.

1.3.36 The domain of $f \circ g$ is the set of all x such that $x^2 + 3 \geq 0$ and $\sqrt{x^2 + 3} - 3 \geq 0$. In other words, $|x| \geq \sqrt{6}$.

$$f(g(x)) = \sqrt{\sqrt{x^2 + 3} + 3}.$$

The domain of $g \circ f$ is the set of all x such that $x \geq 3$ and $(\sqrt{x - 3})^2 + 3 \geq 0$. In other words, $x \geq 3$.

$$g(f(x)) = \sqrt{x}.$$

4. Homework

1.4.6 We are looking for all lines

$$y - y_0 = m(x - x_0)$$

where m is any number and (x_0, y_0) is the intersection point of the lines given. There is also a vertical line $x = x_0$. Therefore, we need to identify (x_0, y_0). From the first line, $y = (5x + 11)/3$. Substituting this into the second line gives $2x - 3(5x + 11) = -7$ or $x = -2$. So, $x_0 = -2$ and

$$(x_0, y_0) = (-2, 1/3).$$

1.5.56 Let α be the angle from the ground to the bottom of the rocket. Then,

$$\cot \alpha = x/b$$

and

$$\cot(\theta + \alpha) = x/(a + b).$$

Therefore,

$$\theta = \cot^{-1}(x/(a + b)) - \alpha = \cot^{-1}(x/(a + b)) - \cot^{-1}(x/b).$$

5. Homework

1.5.40(d) Draw a right triangle with θ as one of the acute angles. Suppose $\tan\theta = x$. This means you can label the side opposite θ with a x and the adjacent side with a 1. Therefore, by the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{x^2+1}$ and

$$\sin(\tan^{-1} x) = \sin\theta = x/\sqrt{x^2+1}.$$

1.6.24 Using the properties of Logarithms, the equation can be rewritten as

$$\ln(4x/2x^6) = 0.$$

Using the fact that $y = e^x$ is the inverse function of $y = \ln x$, the solution is any positive number x which stisfies

$$4x/2x^6 = 1 \rightarrow x = 2^{1/5} \approx 1.15.$$

6. Homework

2.1.4 (a) 1 (b) $-\infty$ (c) Does not exist (d) -2

2.2.36 Use a little algebra first.

$$1/x + 1/x^2 = (x+1)/x^2.$$

If $x < 0$ and very close to zero, then $(x+1)/x^2 > 0$. Therefore,

$$\lim_{x \to 0^-} (1/x + 1/x^2) = \lim_{x \to 0^-} (x+1)/x^2 = +\infty$$

7. Homework

2.3.12 Since the degree of the numerator and denominator are the same, the answer is 5/3.

2.3.18 Inside the cube root the degree of the numerator and the denominator are the same. Therefore, the answer is $(3/2)^{1/3}$.

2.3.20 Since the numerator is the square root of a quadratic and the denominator is linear, or first degree, the answer is $\sqrt{5}/1 = \sqrt{5}$.

2.3.26 Since the degree of the numerator is larger than the degree of the denominator, the answer is either $+\infty$ or $-\infty$. Since the difference between the degrees is odd, the ratio of the leading coefficients is negative and $t \to -\infty$, the answer is $+\infty$.

8. Homework

2.5.24 For part (a), f is continuous everywhere if $f(-3) = \lim_{x\to -3} f(x)$. In other words, k must be chosen so that $0 = k/9$ or $k = 0$. For part (b), the same analysis would imply that $9 = +\infty$ if $k > 0$, $9 = -\infty$ if $k < 0$ and $9 = 0$ if $k = 0$. Since none of these makes sense, there is no value of k which will make f continuous everywhere.

2.6.26

$$\lim_{x\to 0} \frac{\sin(6x)}{\sin(8x)} = \frac{6}{8} \lim_{x\to 0} \frac{\sin(6x)}{6x} \lim_{x\to 0} \frac{8x}{\sin(8x)} = \frac{3}{4}(1)(1) = 4/3.$$

9. Homework

3.1.10 For (a), the average rate of change is

$$\frac{2^3-1^3}{2-1} = 7.$$

For part (b),

$$\lim_{h\to 0} \frac{(1+h)^3-1^3}{h} = \lim_{h\to 0} \frac{3h+3h^2+h^3}{h} = \lim_{h\to 0} 3 + 3h + h^2 = 3.$$

For part (c),

$$\lim_{h\to 0} \frac{(x_0+h)^3-x_0^3}{h} = \lim_{h\to 0} \frac{3x_0^2h+3x_0h^2+h^3}{h} = \lim_{h\to 0} 3x_0^2 + 3x_0h + h^2 = 3x_0^2.$$

For part (d), draw a sketch of $y = x^3$ along with a line that passes through the points $(1, 1)$ and $(2, 8)$ as well as the line tangent to $y = x^3$ at the point $(1, 1)$.

3.2.22 This is similar to part (c) above.

$$\frac{4\pi}{3} \lim_{h\to 0} \frac{(r+h)^3-r^3}{h} = \frac{4\pi}{3} \lim_{h\to 0} \frac{3r^2h+3rh^2+h^3}{h} = \frac{4\pi}{3} \lim_{h\to 0} 3r^2 + 3rh + h^2 = 4\pi r^2.$$

10. Homework

3.3.22 Since $y = x^{-3}+x^{-2}+x^{-1}+1+x+x^2+x^3$, $y' = -3x^{-4}-2x^{-3}-x^{-2}+1+2x+3x^2$ and

$$y'(1) = -3-2-1+1+2+3 = 0.$$

3.3.54 If $a > 0$, the tangent at $x = a$ is given by

$$y - 1/a = -a^{-2}(x - a).$$

Therefore, x and y intercepts are $(2a, 0)$ and $(0, 2/a)$. The distance between these intercepts is

$$\sqrt{4a^2 + 4/a^2} = 2\sqrt{a^2 + a^{-2}}.$$

The distance between either intercept and the point of tangency is

$$\sqrt{a^2 + a^{-2}}.$$

Therefore, the point of tangency bisects the line segment connecting the intercepts.

11. Homework

3.4.4 For (a), $f(x) = x^3 - 1$, so $f'(x) = 3x^2$. For (b),

$$f'(x) = (x+1)(2x-1) + (1)(x^2 - x + 1) = 3x^2.$$

3.4.14 Just use the quotient rule.

$$\left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{(x^2-5)(4)-(4x+1)(2x)}{(x^2-5)^2}\right|_{x=1} = -13/8.$$

12. Homework

3.5.18 There are many ways to do this problem and the final answers will look different, but they are the same. First, notice that

$$\frac{(x^2+1)\cot x}{3-\cos x\csc x} = \frac{x^2+1}{3\tan x-1}.$$

Using this perspective,

$$\frac{dy}{dx} = \frac{(3\tan x-1)(2x)-(x^2+1)(3\sec^2 x)}{(3\tan x-1)^2}.$$

3.6.48 Let $f(x) = y$. Then $f(\pi/4) = 3$. The slope is

$$f'(\pi/4) = 12\cot^3(\pi/4)(-\csc^2\pi/4) = -24.$$

Therefore, the line tangent to $y = f(x)$ at the point $(\pi/4, 3)$ is

$$y - 3 = -24(x - \pi/4).$$

13. Homework

4.2.32 Since

$$y = \left(\frac{x-1}{x+1}\right)^{1/5},$$

it follows that

$$\ln y = \frac{1}{5}\ln\left(\frac{x-1}{x+1}\right) = \frac{1}{5}\ln(x-1) - \frac{1}{5}\ln(x+1).$$

Therefore, differentiating in x yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{5x-5} - \frac{1}{5x+5}.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \left(\frac{1}{5x-5} - \frac{1}{5x+5}\right)\left(\frac{x-1}{x+1}\right)^{1/5}.$$

4.3.52 Differentiating with respect to x yields

$$\frac{1}{\sqrt{1-(xy)^2}}\left(x\frac{dy}{dx} + y\right) = \frac{-1}{\sqrt{1-(x-y)^2}}.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx}\left(\frac{x}{\sqrt{1-(xy)^2}}\right) = \frac{-1}{\sqrt{1-(x-y)^2}} - \frac{y}{\sqrt{1-(xy)^2}}$$

and ultimately

$$\frac{dy}{dx} = \frac{\left(\frac{-1}{\sqrt{1-(x-y)^2}} - \frac{y}{\sqrt{1-(xy)^2}}\right)}{\left(\frac{x}{\sqrt{1-(xy)^2}}\right)}$$

14. Homework

4.4.16 Since the limit is of the form $\frac{\infty}{\infty}$, we can use LR to evaluate it.

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)} = \lim_{x\to 0^+} \frac{\cos x/\sin x}{\sec^2 x/\tan x} = \lim_{x\to 0^+} \cos^2 x = 1.$$

5.1.18 Since

$$f'(x) = \frac{x^2-2}{(x^2+2)^2},$$

$f'(x) = 0$ if $x = \pm\sqrt{2}$. Since $f'(-2) < 0$, $f'(0) > 0$ and $f'(2) < 0$, f is decreasing on the intervals $(-\infty, -\sqrt{2})$ and $(\sqrt{2}, +\infty)$ and increasing on $(-\sqrt{2}, \sqrt{2})$.

Since

$$f''(x) = \frac{2x(x^2-6)}{(x^2+2)^4},$$

$f''(x) = 0$ if $x = 0, \pm\sqrt{6}$. Since $f''(-3) < 0$, $f''(-1) > 0$, $f''(1) < 0$ and $f''(3) > 0$, f is concave down on the intervals $(-\infty, -\sqrt{6})$ and $(0, \sqrt{6})$ and concave up on $(-\sqrt{6}, 0)$ and $(\sqrt{6}, +\infty)$. Moreover, since the concavity changes at $x = 0, \pm\sqrt{6}$, f has inflection points at $x = 0, \pm\sqrt{6}$.

15. Homework

Attention: For most examples, the first derivative test will always be conclusive. The functions will either have relative maximums, relative minimums or neither at all critical points. However, you have to pick points other than the critical points to complete the test. The second derivative test may be inconclusive, but you use the critical points to complete the test. On the next exam and final exam, you will be able to use whichever test you want. However, you will have to use the first derivative test at all critical points where the second derivative test is inconclusive. You can choose to use different tests for each critical point. If $f'(x) = 0$ and $f''(x) = 0$ for any critical points, you will have to use the first derivative test. Otherwise, the second derivative test can be used.

5.2.28 Evaluating both derivatives gives us

$$f'(x) = 4x^3 - 36x^2 = 4x^2(x-9) \quad \textit{and} \quad f''(x) = 12x^2 - 72x = 12x(x-6).$$

It follows that $f'(x) = 0$ if $x = 0$ or $x = 9$.

Since $f'(-1) < 0$, $f'(1) < 0$ and $f'(10) > 0$, the first derivative test states that f has a relative minimum of $f(9)$ at $x = 9$ and f does not have a relative extremum at $x = 0$.

Since $f''(0) = 0$, the second derivative test is inconclusive at $x = 0$. Since $f''(9) > 0$, the second derivative test states that f has a relative minimum of $f(9)$ at $x = 9$.

5.2.30 Evaluating both derivatives gives us

$$f'(x) = (x-3)e^x + e^x = (x-2)e^x \quad \textit{and} \quad f''(x) = (x-2)e^x + e^x = (x-1)e^x.$$

It follows that $f'(x) = 0$ if $x = 2$.

Since $f'(0) < 0$ and $f'(3) > 0$, the first derivative test states that f has a relative minimum of $f(2)$ at $x = 2$.

Since $f''(2) > 0$, the second derivative test states that f has a relative minimum of $f(2)$ at $x = 2$.

5.4.14 Since $f(x) = \sin x - \cos x$, $f'(x) = \cos x + \sin x$. Therefore, $f'(x) = 0$ if $\cos x = -\sin x$. Since $0 \leq x \leq \pi$, $x = 3\pi/4$. Since

$$f(0) = -1, \quad f(\pi) = 1, \quad \textit{and} \quad f(3\pi/4) = \sqrt{2},$$

the absolute maximum is $\sqrt{2}$ when $x = 3\pi/4$ and the absolute minimum is -1 when $x = 0$.

5.4.20 Since $f(x) = x^4 + 4x$ is a fourth degree polynomial with positive leading coefficient,

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty.$$

Therefore, $f(x)$ does not have a maximum value and the minumum value must occur at a stationary point. Notice that

$$f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 0$$

only if $x = -1$. So, the minimum value of $f(x)$ is $f(-1) = -3$.

17. Homework

5.5.38 The area of the trapezoid is

$$A(\theta) = \frac{1}{2}(4 + 4\cos\theta)(2\sin\theta) = 4\sin\theta + 4\cos\theta\sin\theta$$

and $0 \le \theta \le \pi/2$.

Solving

$$0 = A'(\theta) = 4\cos\theta + 4\cos^2\theta - 4\sin^2\theta = 8\cos^2\theta + 4\cos\theta - 4 = 2(2\cos\theta - 1)(\cos\theta + 1)$$

has two solutions in terms of $\cos\theta$. Either $\cos\theta = 1/2$ or $\cos\theta = -1$. Since $0 < \theta < \pi/2$, $\cos\theta = 1/2$ or $\theta = \pi/3$.

$$\cos\theta = 1/2,\ -1.$$

Since

$$A(0) = 0, \quad A(\pi/3) \approx 5.2, \quad A(\pi/2) = 4$$

the maximum area is approximately 5.2 $units^2$.

One should note that bees solved this problem many years ago and they don't know calculus! In case you are not sure what I mean, the trapezoid has the same proportions as half of a honey-comb which allows the bees to store the most honey while using the least material to construct the honey-comb.

5.5.58 Let $1 - x$ be the distance from P to the town. It follows that $0 \le x \le 1$. Recall that distance is equal to rate times time.

For part (a), the time it takes the man to get to town is

$$T(x) = \sqrt{1 + x^2}/3 + (1 - x)/5.$$

If he rows the entire way, it will take $T(1) = \sqrt{2}/3 \approx 0.471$ hours. If he rows straight to shore and then walks it will take$T(0) = 1/3 + 1/5 \approx .53$ hours. Solving

$$0 = T'(x) = x/(3\sqrt{1 + x^2}) - 1/5$$

has $x = 3/4$ as a solution. Since $T(3/4) = 5/12 + 1/20 = 0.467$ hours, the man should row to the point $1 - 3/4 = 1/4$ of a mile from town.

For part (b), $T(x) = \sqrt{1 + x^2}/4 + (1 - x)/5$ and $T'(x) = x/(4\sqrt{1 + x^2}) - 1/5$. Since $T'(x) = 0$ if $x = 4/3$ and $4/3 > 1$, the shortest time occurs if he rows all the way or rows straight to shore and then walks. Since

$$T(0) = 1/4 + 1/5 = .45, \quad T(1) = 1/\sqrt{2} \approx .35$$

he should row straight to town.

5.5.44 (a) $R(x) = x(1000 - x) = 1000x - x^2$
(b) $P(x) = 1000x - x^2 - 3000 - 20x = -x^2 + 980x - 3000$
(c) If $x = 0$, profit will be $P(0) = -3000$. If $x = 500$, profit will be $P(500) = 237000$. Solving $P'(x) = 0$ is the same as solving $-2x + 980 = 0$ or $x = 490$. Since $P(490) = 237100$, the company must produce 490 units to maximize profit.
(d) The maximum profit is $P(490) = \$237100$.
(e) A price of $p = \$1000 - 490 = \$510$ must be charged to obtain the maximum profit.

19. Homework

5.7.4 The function $f(x) = \ln(4 + 2x - x^2)$ is differentiable if $4 + 2x - x^2 > 0$ or $1 - \sqrt{5} < x < 1 + \sqrt{5}$. Since $1 - \sqrt{5} < -1 < 3 < 1 + \sqrt{5}$ and $f(-1) = 0$ and $f(3) = 0$, the hypotheses of Rolle's Theorem are satisfied and there must be a number $-1 < c < 3$ such that $f'(c) = 0$. Since $f'(x) = \frac{2-2x}{4+2x-x^2}$, $f'(x) = 0$ if $x = 1$. Therefore, $c = 1$.

5.8.18 (a)$v(t) = t/2 - 1/(t + 1)$ and $a(t) = 1/2 + 1/(t + 1)^2$.
(b)$s(1) = 1/4 - \ln 2$, $v(1) = 0$ $feet/second$ and $a(1) = 1/2 + 1/4$ $feet/second/second$.
(c)$v(t) = 0$ if $\frac{t(t+1)-2}{2(t+1)} = \frac{t^2+t-2}{2t+2} = \frac{(t+2)(t-1)}{2t+2} = 0$. Therefore, since $t \geq 0$, the particle is stopped when $t = 1$.
(d)Since $a(t) > 0$ for all $t > 0$, the particle is speeding up when $1 < t < \infty$ and slowing down when $0 \leq t < 1$ because $v(t) > 0$ if $t > 1$ and $v(t) < 0$ if $0 \leq t < 1$.
(e) Since $s(t)$ has a relative minimum when $t = 1$, the distance travelled from $t = 0$ to $t = 5$ is $(s(0) - s(1)) + (s(5) - s(1)) \approx 5.3$ $feet$

5.8.16 (a)$v(t) = -\frac{t^2-4}{(t^2+4)^2}$ and $a(t) = \frac{2t(t^2-12)}{(t^2+4)^3}$.
(b)$s(1) = 1/5$, $v(1) = 3/25\, feet/second$ and $a(1) = -22/125\, feet/second/second.$
(c)$v(t) = 0$ if $t = 2$. Therefore, the particle is stopped when $t = 2$.
(d)Notice that $a(t) = 0$ when $t = 0$ and $t = 2\sqrt{3}$. Since $v(t) > 0$ when $t < 2$ and $a(t) > 0$ when $t > 2\sqrt{3}$, the particle is speeding up when $2 < t < 2\sqrt{3}$ and the particle is slowing down when $0 \le t < 2$ and $2\sqrt{3} < t$. (e) Since $s(t)$ has a relaitve maximum when $t = 2$, the distance travelled is equal to $(s(2) - s(0)) + (s(2) - s(5)) = 19/58 \approx 0.33$

5.8.30 The velocity is $v(t) = \frac{-200t}{(t^2+12)^2}$ and the acceleration is $a(t) = \frac{600(t^2-4)}{(t^2+12)^3}$. Since $a(t) = 0$ when $t = 2$ and $v(0) = 0$, $v(2) = -25/16$ and $\lim_{t\to+\infty} v(t) = 0$, the maximum velocity is 0 and the minimum velocity is $-25/16$. Therefore, when $t = 2$, the particles speed is maximum at 25/16 feet per second. Since $v(2) < 0$, the particle is moving down the s-axis when it attains its maximum speed.

21. Homework

6.3.2(a)

$$\int \sec^2(4x+1)\ dx = \tfrac{1}{4}\int \sec^2 u\ du = \tfrac{1}{4}\tan u + C = \tfrac{1}{4}\tan(4x+1) + C$$

6.3.2(b)

$$\int y\sqrt{1+2y^2}\ dy = \tfrac{1}{4}\int \sqrt{u}\ du = \tfrac{1}{6}u^{3/2} + C = \tfrac{1}{6}(1+2y^2)^{3/2} + C$$

6.3.2(c)

$$\int \sqrt{\sin \pi\theta}\cos \pi\theta\ d\theta = \tfrac{1}{\pi}\int \sqrt{u}\ du = \tfrac{2}{3\pi}u^{3/2} + C = \tfrac{2}{3\pi}(\sin \pi\theta)^{3/2} + C$$

6.3.2(d)

$$\int (2x+7)(x^2+7x+3)^{4/5}\ dx = \int u^{4/5}\ du = \tfrac{5}{9}u^{9/5} + C = \tfrac{5}{9}(x^2+7x+3)^{9/5} + C$$

22. Homework

6.3.30 Let $u = t^2$. Then
$$\int \tfrac{t}{t^4+1}\,dt = \tfrac{1}{2}\int \tfrac{1}{u^2+1}\,du = \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1}(t^2) + C.$$

6.8.18 Let $u = e^{-x}$. Then
$$\int_{\ln 2}^{\ln(2/\sqrt{3})} \tfrac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx = -\int_{1/2}^{\sqrt{3}/2} \tfrac{1}{\sqrt{1-u^2}}\,du = \cos^{-1}(\sqrt{3}/2) - \cos^{-1}(1/2) = -\pi/6.$$