SOBOLEV SPACES WITH ONLY TRIVIAL ISOMETRIES

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Abstract. We will give some conditions for Sobolev spaces on bounded Lipschitz domains to admit only trivial isometries.

1. Introduction

The first example of a normed space with trivial isometries was given by Pelczynski as a space of continuous functions on a certain topological space admitting only trivial homeomorphisms (see [6]). The only isometries of these spaces are $\pm I$, where I is the identity. Davis [6] proved that a separable Hilbert space can be renormed so that the new space has only trivial isometries. A result of Plotkin [15], that appeared at about the same time, implies that if U is a domain in $\mathbb{R}^n$, $n > 2$, $p > 0$, $p \notin 2\mathbb{N}$, $p \neq 2n/(n-2)$, and U is not invariant with respect to any non-trivial composition of a translation, reflection and homothety, then the subspace of $L_p(U)$ consisting of harmonic functions has only trivial isometries. Bellenot [3] generalized the result of [6] by proving that a renorming with trivial isometries exists for any real separable Banach space, and Jarosz [11] showed that the same is true without the requirement of separability. Bellenot [4] proved that the (real) James space has only trivial isometries, and Semenov and Skorik [21] established this property of the James space in both real and complex cases. The latter result was extended by Sersouri [20] to some generalizations of the James space. Gordon and Lewis [8] showed that all compact symmetric subgroups of the orthogonal group O_n which contain the permutation group S_n can be realized as the group of all isometries on an appropriate Banach space. Gordon and Loewy [9] showed that if $B = \{b_i\}_{i\geq 1}$ and B_1 are two (Δ) bases for the same Banach space, then $B_1 = \{\epsilon_i b_i\}_{i\geq 1}$ where $\epsilon_i = \pm 1$. Casazza and Shura [5] characterized the group of isometries of Tsirelson's space which appears to be very small and consists of certain sign-changing permutations of coordinates. We refer the reader to the survey [7], p.94-96 for more detailed history.

We will denote by $W_p^k(U)$ the Sobolev space with p-integrability and k-smoothness over the domain U. The space $C^1(\overline{U})$ will be the set of all continuously differentiable functions f, defined on U, such that f and all of its first order partial derivatives are restrictions of continuous functions defined on the closed, and for our purposes compact set $\overline{U}$.

We say a set mapping $\tau : \mathbb{R}^n \to \mathbb{R}^n$ is a composition of a translation and a sign-changing permutation of coordinates if there exists a permutation σ of the set $\{1, 2, ..., n\}$ and constants b_i such that $\tau_i(x) = \pm x_{\sigma(i)} + b_i$ for $i \in \{1, 2, ..., n\}$. We prove the following.

Theorem 1. *Let U be an open, bounded, connected domain in $\mathbb{R}^n$ and suppose U is not invariant with respect to any composition of a translation and sign-changing permutation*

Date: August 23, 2006.

1991 *Mathematics Subject Classification.* Primary 42B20, 42B25. Secondary 46B70, 47B38.

Key words and phrases. Banach Spaces, Isometries, Sobolev Spaces.

of coordinates . Let $1 \le p < \infty$, $p \notin 2\mathbb{N}$, $k \in \mathbb{N}$ *and* E *be the subspace of* $W_p^k(U)$ *consisting of all* $C^1(\overline{U})$ *functions. Then* $T : E \to E$ *is a surjective linear isometry if and only if*

$$T(f)(x) = \pm f(x)$$

Corollary 1. *Let* U *be an open, bounded, connected, Lipschitz domain in* $\mathbb{R}^n$ *which is not invariant with respect to any composition of a translation and sign-changing permutation of coordinates. Let* $1 \le p < \infty$, $p \notin 2\mathbb{N}$, *and* $k \in \mathbb{N}$ *such that* $k > \frac{n}{p} + 1$. *Then* $T : W_p^k(U) \to W_p^k(U)$ *is a surjective linear isometry if and only if*

$$T(f)(x) = \pm f(x).$$

2. Sobolev spaces

In this paper, we will consider Sobolev spaces over bounded domains and having integer smoothness. For an open subset U of $\mathbb{R}^n$, $1 \le p < \infty$ and $k \in \mathbb{N}$, we consider the Sobolev space $W_p^k(U)$ of distributions with all derivatives up to and including order k in $L^p(U)$. For $f \in W_p^k(U)$,

$$\|f\|_{p,k} = \left(\sum_{0 \le |\alpha| \le k} \|D^\alpha f\|^p_{L_p(U)} \right)^{1/p},$$

where, for each multiindex $\alpha = (\alpha_1, ..., \alpha_n)$, we denote by

$$|\alpha| = \alpha_1 + ... + \alpha_n,$$

and

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}(f).$$

We will also make special assumptions on the boundary of the underlying domains. Specifically, we will consider domains whose boundaries satisfy the strong local Lipschitz condition. We will refer to such domains as Lipschitz domains. We refer the reader to [1] for a complete discussion of the strong local Lipschitz condition, Sobolev imbedding theorems, and Sobolev spaces in general. For Lipschitz domains, our main result will apply to operators defined on the entire Sobolev space W_p^k. This is done by using one of the Sobolev embedding theorems. For our purposes, if U is Lipschitz and p, q, and n satisfy $k > \frac{n}{p} + 1$, each element of $W_p^k(U)$ is almost everywhere equal to a function in $C^{1,\gamma}(\overline{U})$. For $0 < \gamma \le 1$, $C^{1,\gamma}(\overline{U})$ consists of all functions f in $C^1(\overline{U})$ such that f and all of its first order partial derivatives are Hölder continuous.

3. Main theorem

Let U_1 and U_2 be open, bounded, connected subsets of $\mathbb{R}^n$. Denote by E_i the subspace of $W_p^k(U_i)$ consisting of all $C^1(\overline{U_i})$ functions. Here, $C^1(\overline{U_i})$ is the set of all functions f such that f and all of its first-order partial deriviatives are the restrictions of functions from $C(\overline{U_i})$. Our main result is

Theorem 2. *Let* U_1 *and* U_2 *be open, bounded, connected, domains in* $\mathbb{R}^n$. *Let* $1 \le p < \infty$, $p \notin 2\mathbb{N}$, *and* $k \in \mathbb{N}$. *Then* $T : E_1 \to E_2$ *is a surjective linear isometry if and only if*

$$T(f)(x) = \pm f(\tau(x))$$

where τ *is a composition of a translation and a sign-changing permutation of coordinates.*

Corollary 2. *Let U_1 and U_2 be open, bounded, connected, Lipschitz domains in $\mathbb{R}^n$. Let $1 \leq p < \infty$, $p \notin 2\mathbb{N}$, and $k \in \mathbb{N}$ such that $k > \frac{n}{p} + 1$. Then $T : W_p^k(U_1) \to W_p^k(U_2)$ is a surjective linear isometry if and only if*

$$T(f)(x) = \pm f(\tau(x)),$$

where τ is a composition of a translation and a sign-changing permutation of coordinates.

Theorem 1 follows from Theorem 2, while Corollary 2 and Corollary 1 follow from Theorem 2 and the Sobolev imbedding theorems [1].

It is still uncertain if any of these results hold for even p. Let $\epsilon > 0$ be small. If it were known that, for all surjective linear operators $T : W_p^k(U) \to W_p^k(U)$ with norm equal to $1 + \epsilon$, there existed a linear isometry $I : W_p^k(U) \to W_p^k(U)$ such that $\|T - I\| < \epsilon$, then Corollary 2 and Corollary 1 could be extended to even p and to $k \geq \frac{n}{p} + 1$. The following example would then show that the estimate on k would be sharp.

For $m \in \mathbb{N} \cup \{0\}$, the following counterexample shows that when $k = \frac{n}{p} - m$, τ need not be a composition of a translation and a sign-changing permutation of coordinates. Let U be any open connected subset of $\mathbb{R}^n$, $k = 1$, $p = 2$, and $n = 2(m + 1)$. Also, let A be a block $n \times n$ matrix that rotates the first 2 dimensions by an angle $\theta \in [0, 2\pi)$ and is a sign-changing permutation of coordinates in the remaining dimensions. Since $p = 2$ and $k = 1$, it is easy to see that the operator $T : W_2^1(AU) \to W_2^1(U)$ defined by $Tf(x) = f(Ax)$ is an isometry. Moreover, any such operator fails to be an isometry when $k > 1$, unless the first block of A is a matrix corresponding to a sign-changing permutation of coordinates on $\mathbb{R}^2$.

It remains unknown if Corollary 2 and Corollary 1 can be extended to $p \neq 2$ and $k < \frac{n}{p}+1$. Also, it is still unknown whether the domains need to be bounded or Lipschitz.

Since the sufficiency in Theorem 2 is obvious, we will assume, for the remainder of the paper, that T is a surjective linear isometry.

4. Isometric Embeddings of W_p^k and the Extension Theorem

Let $1 \leq p < \infty$ and $k \in \mathbb{N}$ we start by constructing isometric embeddings of the spaces W_p^k into certain L_p-spaces. Let U_1 and U_2 be open bounded connected subsets of $\mathbb{R}^n$. For each multiindex α, consider translations

$$U_i^\alpha = U_i + a\alpha$$

where the real number a is chosen so that $U_i^\alpha \cap U_i^\beta = \emptyset$ when $\alpha \neq \beta$. Notice if $\alpha = (0, ..., 0)$, U_i^α is just U_i. Denote by

$$U_i' = \bigcup_{|\alpha| \leq k} U_i^\alpha.$$

We define isometric embeddings

$$T_i : W_p^k(U_i) \mapsto L_p(U_i')$$

as follows: for $f \in W_p^k(U_i)$ put $T_i f(x) = D^\alpha f(x - a\alpha)$ if $x \in U_i^\alpha$.

Since $T : E_1 \to E_2$ is a surjective linear isometry,

$$S = T_2 T T_1^{-1}$$

is an isometric operator from $T_1(E_1) \subset L_p(U_1')$ onto $T_2(E_2) \subset L_p(U_2')$.

As in the paper of Plotkin [15] mentioned in the Introduction, our result is based on the extension theorem for L_p-isometries proved by Plotkin [14]-[18] and, later and independently, by Hardin [10] and Rudin [19]. One should note that everything in this section is done under the assumption that $p \notin 2\mathbb{N}$.

Theorem 3. *(Extension Theorem) Let $p > 0$, $p \notin 2\mathbb{N}$, (Ω_1, σ_1) and (Ω_2, σ_2) be spaces with finite measures, H be a subspace of $L_p(\Omega_1, \sigma_1)$ containing the constant function $1(\omega) = 1$, and $T : H \to L_p(\Omega_2, \sigma_2)$ be a linear isometry. Then there exists a linear isometry $T' : L_p(\Omega_1, \mathcal{A}, \sigma_1) \to L_p(\Omega_2, \sigma_2)$ so that $T'|_H = T$, where $\mathcal{A}$ is the smallest σ-algebra of subsets of Ω_1 making all functions from H measurable.*

The survey [12] contains a short proof of the extension theorem and references to different applications and generalizations. Now, we need a simple fact whose proof we leave to the reader.

Lemma 1. *Let U be a open bounded set in $\mathbb{R}^n$, and H be a set of continuous functions on U separating the points of U, i.e. for every $x, y \in U$, $x \neq y$ there exists $f \in H$ so that $f(x) \neq f(y)$. Then the smallest σ-algebra making all functions from H measurable is the σ-algebra of all Borel subsets of U.*

Proposition 1. *Let T, T_i, and S be as above and let $H = T_1(E_1)$. If $p \notin 2\mathbb{N}$ then there exists an isometry S' from $L_p(U_1')$ into $L_p(U_2')$ so that $S'|_H = S$.*

Proof. Consider the function $f_0 \in W_p^k(U)$ where

$$f_0(x) = exp(x_1 + x_2 + ... + x_n).$$

Since U_1 is bounded and the derivatives of f_0 are equal to f_0, the function $T_1(f_0)$ is bounded and separated from zero on U'. Therefore, the operator $Rf = f/T_1(f_0)$ is an isometry from $L_p(U_1')$ onto $L_p(U_1', |T_1(f_0)|^p dx)$. Now, $R(H)$ is a subspace of $L_p(U_1', |T_1(f_0)|^p dx)$ containing the constant functions.

Let us prove that $R(H)$ contains a set of continuous functions that separate the points of U_1'. Consider arbitrary $y, z \in U_1'$, $y \neq z$. There exists $1 \leq i \leq n$ so that $y_i \neq z_i$. If $y, z \in U_1^\alpha$ then y, z are separated by any function $R(T_1(f)) \in R(H)$ where $D^\alpha f(x) = x_i exp(x_1 + ... + x_n)$. If $y \in U_1^\alpha$ and $z \in U_1^\beta$ such that $\alpha_j < \beta_j$ for some j, then $f(x) = x^\alpha (= x_1^{\alpha_1} ... x_n^{\alpha_n})$. Otherwise, let $f(x) = x^\beta$.

By Lemma 1, the smallest σ-algebra of subsets of U_1' making all functions from $R(H)$ measurable is the σ-algebra of all Borel sets. Applying the Extension Theorem to the isometric operator SR^{-1} mapping $R(H) \subset L_p(U_1', |T_1(f_0)|^p dx)$ into $L_p(U_2')$, we get an isometric operator $P : L_p(U_1', |T_1(f_0)|^p dx) \to L_p(U_2')$ such that $P|_{R(H)} = SR^{-1}$. Then the operator $S' = PR$ is an isometry from $L_p(U_1')$ into $L_p(U_2')$ such that $S'|_H = S$ and $S'(H) = T_2(E_2)$. □

5. Lamperti's Theorem and the mapping τ

A regular set isomorphism of $\mathbb{R}^n$ with respect to Lebesgue measure will mean a mapping ϕ of the Borel sets into the Borel sets defined modulo sets of measure zero, such that

$$\phi(\mathbb{R}^n \setminus U) = \phi(\mathbb{R}^n) \setminus \phi(U) \tag{5.1}$$

$$\phi(\bigcup_{i=1}^{\infty} U_i) = \bigcup_{i=1}^{\infty} \phi(U_i) \text{ for disjoint } \mathrm{U_i} \tag{5.2}$$

$$|\phi(U)| = 0 \text{ if, and only if, } |\text{U}| = 0 \tag{5.3}$$

The classical result of Lamperti [13] provides a characterization of injective isometries between two L_p-spaces. Applied to the operator S', it shows that there exists a function $F \in L_p(U_2')$ and a regular set isomorphism ϕ so that $S'(f) = F\phi(f)$ for all simple function f defined on $L^p(U_1')$. Since the simple functions are dense in L^p, ϕ can also be thought of as an isometric homomorphism (with respect to multiplication) $\phi : L_\infty(U_1') \to L_\infty(U_2')$ so that $S'f = F\phi(f)$ for every $f \in L_\infty(U_1')$. In particular, by (5.3), given a set U, $\phi(\chi_U) = 0$ a.e.(almost everywhere with respect to Lebesgue measure) if and only if $|U| = 0$. Throughout this paper, χ_U will denote the characteristic function of a set U.

Consider the functions $f_i(x) = x_i$, for all $i \in \{1, 2, ..., n\}$ Since $U_1^{'}$ is bounded, we have $f_i \in L_\infty(U_1^{'})$. Let $\tau_i = \phi(f_i) \in L_\infty(U_2^{'})$, and define a mapping $\tau : U_2^{'} \mapsto \mathbb{R}^n$ by $\tau(x) = (\tau_1(x), ..., \tau_n(x))$. This mapping has the following properties.

Lemma 2. $S^{'}T_1p = FT_1p(\tau)$ *for every polynomial* $p \in C^\infty(\overline{U_1})$.

Proof. Using the result of Lamperti mentioned earlier, for every polynomial $p \in C^\infty(\overline{U_1})$, T_1p is a polynomial on each component of $U_1^{'}$. Therefore, T_1p can be written as a finite sum of polynomials

$$q(x_1, ..., x_n) = \sum_{j=1}^{m} a_j \prod_{i=1}^{n} x_i^{r_{i,j}}.$$

For each of these we have

$$\begin{aligned} S^{'}q &= \sum_{j=1}^{m} a_j S^{'}(\prod_{i=1}^{n} x_i^{r_{i,j}}) = F(x) \sum_{j=1}^{m} a_j \phi(\prod_{i=1}^{n} x_i^{r_{i,j}}) \\ &= F(x) \sum_{j=1}^{m} a_j \prod_{i=1}^{n} \phi(x_i)^{r_{i,j}} = F(x) \sum_{j=1}^{m} a_j \prod_{i=1}^{n} \tau_i(x)^{r_{i,j}} \\ &= F(x) q(\tau(x)). \end{aligned}$$

□

Lemma 3. $\tau(U_2^{'}) \subset U_1^{'}$

Proof. Suppose that there exists $x_0 \in U_2^{'}$ for which $\tau(x_0) \notin U_1^{'}$. Consider a polynomial $Q(x) = A - \sum_{i=1}^{n}(x_i - \tau_i(x_0))^2$ where we choose $A > 0$ so that Q is positive on $U_1^{'}$. Then,

$$A = Q(\tau(x_0)) = \sup_{U_2'} Q(\tau) = \|\phi(Q)\|_{L^\infty(U_2')} = \|Q\|_{L^\infty(U_1')} = \sup_{U_1'} Q < A,$$

and we get a contradiction. □

Let $f \in C(\overline{U_1'})$ and $\{q_j\}_{j=1}^{\infty}$ be a sequence of polynomials converging to f in the supremum norm. Since $\tau(U_2^{'}) \subset U_1^{'}$ and $F \in L^p(U_2^{'})$, we have

$$\begin{aligned} &\lim_{j\to\infty} \int_{U_2'} |F(x)q_j(\tau(x)) - F(x)f(\tau(x))|^p dx \\ &\qquad \le \lim_{j\to\infty} \sup_{U_1'} |q_j - f|^p \|F\|_p^p \\ &\qquad = 0 \end{aligned}$$

This means that $S'(f)(x) = F(x)f(\tau(x))$ for all $f \in C(\overline{U_1'})$. In particular, this is true for all functions in $T_1(C^\infty(\overline{U_1}) \cap W_p^k(U_1))$. Since U_1 is Lipschitz, $C^\infty(\overline{U_1})$ is dense in $W_p^k(U_1)$ and $S'(f)(x) = F(x)f(\tau(x))$ for all $f \in T_1(E_1)$.

6. Proof of Theorem 2

Proof. Since $T_1(\chi_{U_1}) \in C(\overline{U_1'})$,

$$S'T_1(\chi_{U_1}) = T_2(F\chi_{\tau^{-1}(U_1)\cap U_2}) \in T_2(E_2).$$

By continuity, $\tau^{-1}(U_1) \cap U_2$ has a nonempty interior. If $\tau^{-1}(U_1) \cap U_2$ is a proper subset of U_2, there exists a sequence $(x_m)_{m=1}^\infty$ in the interior of $\tau^{-1}(U_1) \cap U_2$ such that

$$\lim_{m\to\infty} F(x_m) = 0.$$

So,

$$\lim_{m\to\infty} |F(x_m)f(\tau(x_m))| \leq \lim_{m\to\infty} |F(x_m)| \max_{\overline{U_1}} |f| = 0$$

for all $f \in T_1(E_1)$. Therefore, since $E_2 = T(E_1)$,

$$\lim_{m\to\infty} g(x_m) = 0$$

for all $g \in T_2(E_2)$. This obvious contradiction implies that $\tau^{-1}(U_1) \cap U_2 = U_2$ and $F \neq 0$ on U_2.

Since

$$S'T_1(x_j) = T_2(F\tau_j),$$

$\tau_j \in C^1(\overline{U_2})$ for all $1 \leq j \leq n$. This means that τ is a continuously differentiable mapping and therefore $\tau(U_2)$ is a connected subset of U_1. Since T is surjective, there exists an $f \in E_1$ such that $T(f) = \chi_{U_2}$. Therefore,

$$S'T_1(f) = T_2(\chi_{U_2}) = \chi_{U_2}$$

and $T_1(f)$ must be supported in U_1. If $T_1(f)$ was supported in a larger subset of U_1' than U_1, then $S'T_1(f)$ would be supported in a larger subset of U_2' than U_2. This follows from the fact that the set function ϕ, and hence τ maps disjoint sets to disjoint sets modulo null sets. So, there exists a nonzero constant C such that $T(\chi_{U_1}) = C\chi_{U_2}$. Moreover, τ is a continuously differentiable invertible mapping from U_2 onto U_1 such that

$$T(f) = Cf(\tau)$$

for all $f \in E_1$. Also, the Jacobian of τ, J_τ, satisfies

$$|detJ_\tau| = |C|^p$$

and therefore

$$\int_{U_1} |f(x)|^p dx = \int_{U_2} |f(\tau(x))|^p |C|^p dx$$

by the change of variables $x \mapsto \tau(x)$.

Since $T(x_1) = C\tau_1(x)$, $CD^\alpha\tau_1(x - a\alpha) = F(x)\chi_{\tau^{-1}(U_1+ae_1)\cap U_2^\alpha}$, which is in $C(\overline{U_2^\alpha})$ for all $|\alpha| = 1$. Here, $e_1 = (1, 0, ..., 0)$. Since $T(x_1)$ is not a constant function, there must exist an α, with $|\alpha| = 1$, such that $\tau^{-1}(U_1 + ae_1) \cap U_2^\alpha$ has nonempty interior. Because $E_i \subset C^1(\overline{U_i})$, we can argue as above that $\tau^{-1}(U_1 + ae_1) \cap U_2^\alpha = U_2^\alpha$. If not, we can show that

$$\lim_{m\to\infty} D^\alpha g(x_m - a\alpha) = 0$$

for all $g \in E_2$. Since $\alpha = e_j$ for some $j \in \{1, 2, ..., n\}$ and T is surjective, there exists $f \in E_1$ such that $T(f) = x_j$. Since $\tau(U_2 + ae_j) \subset U_1 + ae_1$, $T_1(f)$ must be supported in $U_1 \cup (U_1 + ae_1)$. Therefore, T maps linear functions of x_1 to linear functions of x_j. Moreover, $S'T_1(x_2)$ cannot be supported in $U_2 + ae_j$. Since $T(x_2)$ is not the constant function, there must exist another $\alpha \neq e_j$ with $|\alpha| = 1$ such that $\tau^{-1}(U_1 + ae_2) \cap U_2^\alpha$ has nonempty interior. Repeating the above arguments one can see that T maps linear functions of x_2 to linear functions of x_i, where $i \neq j$. Continuing inductively, there exists a permutation σ of the set $\{1, 2, ..., n\}$ such that

$$T(x_j) = C(m_j x_{\sigma(j)} + b_j)$$

for constants $m_j \neq 0$ and b_j.

By using the change of variables $x \mapsto \tau(x)$ and the fact that $\|T\| = 1$, we have

$$\int_{U_1} |x_j|^p = \int_{U_2} |C|^p |\tau_j(x)|^p$$

and

$$\int_{U_1} = \int_{U_2} |C|^p |m_j|^p.$$

Since we already know that

$$\int_{U_1} = \int_{U_2} |C|^p,$$

$|m_j| = 1$ and hence $|C|^p = |det J_\tau(x)| = 1$. Therefore,

$$T(f) = \pm f(\tau)$$

where τ is a composition of a translation and a sign-changing permutation of coordinates. □

Notice that all the arguments above are independent of real or complex-valued functions except for the conclusions. In the complex case, $T(f) = \omega f(\tau)$ where ω is a modulus one complex number.

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