SOBOLEV SPACES WITH ONLY TRIVIAL ISOMETRIES, II

GEOFF DIESTEL

ABSTRACT. In this article, we obtain a canonical form for surjective linear isometries $T : W_p^k(U) \to W_p^k(U)$ provided U is an open, bounded, connected, domain with Lipschitz boundary, $1 \leq p < \infty$, $p \neq 2$ and $T[C(\overline{U})] = C(\overline{U})$. We will show there exists $|c| = 1$ and mapping τ that is a composition of a translation and a sign-changing permutation of coordinates such that $Tf = cf(\tau)$. As a corollary, if $k > \frac{n}{p}$, all surjective isometries $T : W_p^k(U) \to W_p^k(U)$ have this trivial form by the Sobolev Imbedding Theorem.

1. INTRODUCTION

Let $1 \leq p < \infty$ and $k, n \in \mathbb{N}$. If U is an open, bounded and connected domain in $\mathbb{R}^n$, we define the Sobolev Space $W_p^k(U)$ to be the space distributions identified by functions $f : U \to \mathbb{C}$ in $L^p(U)$ such that

$$\|f\|_{k,p}^p = \sum_{|\alpha| \leq k} \|D^\alpha f\|_p^p < \infty.$$

Here and throughout the remainder of the article we use the convention $D^\alpha f = f$ if α is the zero multi-index. Also, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For $1 < p < \infty$ let $(W_p^k(U))^*$ denote the dual of $W_p^k(U)$. Each $L \in (W_p^k(U))^*$ can be identified with a unique $v = (v_\alpha)_{|\alpha| \leq k}$ such that each $v_\alpha \in L_p(U)$ and

$$L(f) = \sum_{|\alpha| \leq k} \int_U D^\alpha f(x) \overline{v_\alpha(x)}\, dx. \tag{1.1}$$

$$\|L\|_{k,p,*}^p = \sum_{|\alpha| \leq k} \|v_\alpha\|_p^p \tag{1.2}$$

For $f, g \in W_p^k(U)$, let $L_f \in (W_p^k(U))^*$ be the support functional of f identified with v_f such that $(v_f)_\alpha = \|f\|_{k,p}^{2-p} |D^\alpha f|^{p-2} \overline{D^\alpha f}$ for all $|\alpha| \leq k$ and

$$L_f(f) = \|f\|_{k,p}^2 \tag{1.3}$$

$$L_f(g) = \|f\|_{k,p}^{2-p} \sum_{|\alpha| \leq k} \int_U D^\alpha g(x) |D^\alpha f(x)|^{p-2} \overline{D^\alpha f(x)}\, dx \tag{1.4}$$

Since $1 < p < \infty$, $[g, f] = L_f(g)$ defines the unique semi-inner product compatible with $\|\cdot\|_{k,p}$.

Let $C(\overline{U})$ be the set of continuous functions $f : U \to \mathbb{C}$ which have continuous extensions to the boundary of U. Let $C_{\mathbb{R}}(\overline{U})$ be the Banach space of continuous functions, $f : \overline{U} \to \mathbb{R}$,

---

Date: March 11, 2008.

1991 *Mathematics Subject Classification.* Primary 42B20, 42B25. Secondary 46B70, 47B38.

Key words and phrases. Banach Spaces, Isometries, Sobolev Spaces.

equipped with the L_∞ norm. By the Banach-Stone Theorem, $J : C_{\mathbb{R}}(\overline{U}) \to C_{\mathbb{R}}(\overline{U})$ is a surjective linear isometry if and only if

$$Jf = hf(\gamma) \tag{1.5}$$

where $|h(x)| = 1$ and $\gamma : \overline{U} \to \overline{U}$ is a homeomorphism.

A regular set isomorphism of $\mathbb{R}^n$ with respect to Lebesgue measure will mean a mapping ϕ of the Borel sets into the Borel sets defined modulo sets of measure zero, such that

$$\phi(\mathbb{R}^n \setminus B) = \phi(\mathbb{R}^n) \setminus \phi(B) \tag{1.6}$$

$$\phi(\bigcup_{i=1}^{\infty} B_i) = \bigcup_{i=1}^{\infty} \phi(B_i) \text{ for disjoint } B_i \tag{1.7}$$

$$|\phi(B)| = 0 \text{ if, and only if, } |B| = 0 \tag{1.8}$$

The classical result of Lamperti [8] provides a characterization of injective isometries between two L_p-spaces.

Lamperti's Theorem. *Let* $(\Omega_i, \Sigma_i, \mu_i)$ *be* σ*-finite measure spaces where* $1 \leq p \neq 2 < \infty$. *Suppose* V *is a linear isometry from* $L_p(\Omega_1, \Sigma_1, \mu_1)$ *into* $L_p(\Omega_2, \Sigma_2, \mu_2)$. *Then there exists a regular set isomorphism* ϕ *from* Σ_1 *into* Σ_2 *and a function* F *defined on* Ω_2 *so that*

$$V(f)(x) = F(x)\gamma(f)(x)$$

where γ *denotes a transformation induced by the set mapping* ϕ *and* F *satisfies*

$$\int_{\phi(A)} |F|^p \, d\mu_2 = \int_{\phi(A)} \tfrac{d(\mu_1 \circ \gamma^{-1})}{d\mu_2} \, d\mu_2 = \mu_1(A)$$

for each $A \in \Sigma_1$. *Conversely, if such a* ϕ *and* F *exist and*

$$V(f)(x) = F(x)\gamma(f)(x)$$

V *is an isometry.*

Since γ is induced by ϕ via $\gamma(\chi_A) = \chi_{\phi(A)}$ for all measureable sets $A \subset \Sigma_1$, properties (1.6), (1.7) and (1.8) imply that γ is an isometric homeomorphism with respect to multiplication on the set of L^∞ functions with compact support.

We say a set mapping $\tau : \mathbb{R}^n \to \mathbb{R}^n$ is a composition of a translation and a sign-changing permutation of coordinates if there exists a permutation σ of the set $\{1, 2, ..., n\}$ and constants $b_i \in \mathbb{R}$ such that

$$\tau_i(x) = \pm x_{\sigma(i)} + b_i$$

for $i \in \{1, 2, ..., n\}$. We will call such a mappings *trivial.* The following are the main results of this article.

Theorem 1. *Let* U *be an open, bounded, connected domain in* $\mathbb{R}^n$ *with Lipschitz boundary,* $1 \leq p \neq 2 < \infty$, $k \in \mathbb{N}$ *and* E *be the subspace of* $W_p^k(U)$ *consisting of all* $C(\overline{U})$ *functions. Then,* $T : W_p^k(U) \to W_p^k(U)$ *is a surjective linear isometry such that* $T[E] = E$ *if and only if there exists a constant* $|c| = 1$ *and a trivial mapping* τ *such that*

$$T(f)(x) = cf(\tau(x)).$$

Corollary 1. *Let* U *be an open, bounded, connected domain in* $\mathbb{R}^n$ *with Lipschitz boundary,* $1 \leq p \neq 2 < \infty$, $k \in \mathbb{N}$ *and* $k > \frac{n}{p}$. *Then* $T : W_p^k(U) \to W_p^k(U)$ *is a surjective linear isometry if and only if there exists a constant* $|c| = 1$ *and a trivial mapping* τ *such that*

$$T(f)(x) = cf(\tau(x)).$$

Corollary 1 is a direct consequence of Theorem 1 and the Sobolev Imbedding Theorem(see [1]).

As in [3], we use the Extension Theorem of Plotkin [9], [10], [11], Hardin [5] and Rudin [12].

Extension Theorem. *Let* $p > 0$, $p \notin 2\mathbb{N}$, (Ω_1, σ_1) *and* (Ω_2, σ_2) *be spaces with finite measures,* H *be a subspace of* $L_p(\Omega_1, \sigma_1)$ *containing the constant function* $1(\omega) = 1$ *and* $T : H \to L_p(\Omega_2, \sigma_2)$ *be a linear isometry. Then there exists a linear isometry* $T' : L_p(\Omega_1, \mathcal{A}, \sigma_1) \to L_p(\Omega_2, \sigma_2)$ *so that* $T'|_H = T$, *where* $\mathcal{A}$ *is the smallest* σ*-algebra of subsets of* Ω_1 *making all functions from* H *measurable.*

There are nice presentations of the Extension Theorem in [7] and [4]. In our proof, we will use the Extension Theorem in two different circumstances. In each we will need $\mathcal{A}$ to be the σ-algebra of Borel sets. In both circumstances we will be able to prove this using the following well known lemma.

Lemma. *Let* Ω *be a open bounded set in* $\mathbb{R}^n$, *and* H *be a set of continuous functions on* Ω *separating the points of* Ω, *i.e. for every* $x, y \in \Omega$, $x \neq y$ *there exists* $f \in H$ *so that* $f(x) \neq f(y)$. *Then the smallest* σ*-algebra making all functions from* H *measurable is the* σ*-algebra of all Borel subsets of* Ω.

In the remainder of this article we prove Theorem 1 which improves the results in [3] by replacing the $C^1(\overline{U})$ condition with an analogous $C(\overline{U})$ condition and replacing the condition $p \notin 2\mathbb{N}$ with $p \neq 2$. It is unclear if the continuity assumption is necessary. However, it is used at key steps throughout our proof. The sufficiecy of Theorem 1 is trivial. In the rest of the article, we prove the necessity.

2. Isometric Embeddings and Extensions

Pick $a > 0$ such that

$$U' = \bigcup_{|\alpha| \leq k} U_\alpha = \bigcup_{|\alpha| \leq k} (U + a\alpha)$$

is a disjoint union. The goal of this section will be to construct isomtries of $L_p(U')$ and $L_{p'}(U')$ which factorize through $W_p^k(U)$ and $(W_p^k(U))^*$ via T and T^*, repectively. Depending on the value of p, the Extension Theorem will guarantee the extension of one of these isomtries. Once this is done, we will be able to apply Lamperti's Theorem to the particular extension.

Recall that $E = W_p^k(U) \cap C(\overline{U})$. Let E^* be the set of support functionals $L_f \in (W_p^k(U))^*$ such that $f \in E$. Define embeddings $I_1 : W_p^k(U) \to L_p(U')$ and $I_2 : (W_p^k(U))^* \to L_{p'}(U')$ by

$$I_1(f)(x) = \sum_{|\alpha| \leq k} D^\alpha f(x - a\alpha) 1_{U_\alpha}(x) \tag{2.9}$$

$$I_2(L)(x) = \sum_{|\alpha| \leq k} v_\alpha(x - a\alpha) 1_{U_\alpha}(x). \tag{2.10}$$

Since we will only use I_2 if $p \in 2\mathbb{N} \setminus \{2\}$, I_2 is well defined by (1.1). By construction, I_1 and I_2 are isometric.

Now, define isometries $S_1 : I_1[W_p^k(U)] \to I_1[W_p^k(U)]$ and $S_2 : I_2[(W_p^k(U))^*] \to I_2[(W_p^k(U))^*]$ by $S_1 = I_1^{-1}TI_1$ and $S_2 = I_2^{-1}T^*I_2$. It is important to notice that $S_1[I_1[E]] = I_1[E]$ and $S_2[I_2[E^*]] = I_2[E^*]$.

Let $f_0(x) = e^{x_1+\cdots+x_n} \in W_p^k(U)$. Define $g_1 = I_1(f_0)$ and $g_2 = I_2(L_{f_0})$. Since f_0 is smooth, positive and invariant under differentiation, we can define isometries $R_1 : I_1[E] \to L_p(U', |g_1|^p\ dx)$ and $R_2 : I_2[E] \to L_{p'}(U', |g_2|^{p'}\ dx)$ by $R_1(f) = f/g_1$ and $R_2(f) = f/g_2$. Notice that $R_1(I_1(f_0)) = R_2(I_2(L_{f_0})) = 1_{U'}$. Let $H_1 = R_1[I_1[E]]$ and $H_2 = R_2[I_2[E^*]]$. Since $C_0^\infty(U) \subset W_p^k(U)$, both H_1 and H_2 contain constant functions and a set of continuous functions that separate points of U'. We refer the reader to Proposition 1 in [3] for details in the case $p \notin 2\mathbb{N}$. The argument for the remaining $p's$ can be easily adapted from the proof of Proposition 1 in [3] by using (1.4). Therefore, using the Extension Theorem, one of the isometries $P_i = R_i^{-1}S_i$ can be extended. In particular, we have that one of S_1 or S_2 can be extended to all of $L_p(U')$ or $L_{p'}(U')$, respectively. When they exist, we will denote these extensions by S_1 and S_2 as well.

3. LAMPERTI'S THEOREM AND THE MAPPING τ, I

In this section we will complete the proof for $p \notin 2\mathbb{N}$. Appying Lamperti's Theorem to the operator S_1 implies that there exists a function $F \in L_p(U')$ and a regular set isomorphism ϕ, which induces a map γ, so that $S_1(f)(x) = F(x)\gamma(f)(x)$ for all $f \in L^p(U')$. Recall that γ can also be thought of as an isometric homomorphism (with respect to multiplication) $\phi, \gamma : L_\infty(U') \to L_\infty(U')$ so that $S_1(fg)(x) = F(x)\gamma(f)(x)\gamma(g)(x)$ for every $f, g \in L_\infty(U')$.

Consider the functions $f_i(x) = x_i 1_{U'}$, for all $i \in \{1, 2, ..., n\}$ Since U' is bounded, we have $f_i \in L_\infty(U')$. Let $\tau_i = \gamma(f_i) \in L_\infty(U')$, and define a mapping $\tau : U' \mapsto \mathbb{R}^n$ by $\tau(x) = (\tau_1(x), ..., \tau_n(x))$. This mapping has the following properties.

Lemma 1. *Let q be a finite linear combination of functions $p_\alpha 1_{U_\alpha}$, where p_α are polynomials. Then $S_1(q)(x) = F(x)q(\tau(x))$.*

Proof. Since q is a finite linear combination, assume q is one of its components.

$$q(x_1, ..., x_n) = \sum_{j=1}^{m} a_j \prod_{i=1}^{n} x_i^{r_{i,j}}.$$

Since ϕ, and hence γ, is a multiplicative homeomorphism,

$$\begin{aligned} S_1(q) &= \sum_{j=1}^{m} a_j S_1\big(\prod_{i=1}^{n} x_i^{r_{i,j}}\big) = F(x)\sum_{j=1}^{m} a_j \gamma\big(\prod_{i=1}^{n} x_i^{r_{i,j}}\big) \\ &= F(x)\sum_{j=1}^{m} a_j \prod_{i=1}^{n} \gamma(x_i)^{r_{i,j}} = F(x)\sum_{j=1}^{m} a_j \prod_{i=1}^{n} \tau_i(x)^{r_{i,j}} \\ &= F(x)q(\tau(x)). \end{aligned}$$

□

Lemma 2. *The point mapping τ satisfies $\tau(U') \subset U'$*

Proof. Suppose that there exists $x_0 \in U'$ for which $\tau(x_0) \notin U'$. Consider a polynomial $Q(x) = A - \sum_{i=1}^{n}(x_i - \tau_i(x_0))^2$ where we choose $A > 0$ so that Q is positive on U'. Then,

$$\begin{aligned} A &= Q(\tau(x_0)) = \sup_{U'} Q(\tau) \\ &= \|\phi(Q)\|_{L^\infty(U')} = \|Q\|_{L^\infty(U')} \\ &< A, \end{aligned}$$

and we get a contradiction. ☐

Let $f \in C(\overline{U'})$. Since f is the uniform limit of functions described in Lemma 1, $S_1(f)(x) = F(x)f(\tau(x))$. Since $1_U = I_1(1_U) \in C(\overline{U'})$,

$$S_1(I_1(1_U)) = I_1(F 1_{\tau^{-1}(U)\cap U}) \in I_1[E].$$

By continuity, $\tau^{-1}(U) \cap U$ has nonempty interior. If $\tau^{-1}(U) \cap U$ is a proper subset of U, there exists a sequence $(x_m)_{m=1}^{\infty}$ in $\tau^{-1}(U) \cap U$ such that

$$\lim_{m\to\infty} F(x_m) = 0.$$

So,

$$\lim_{m\to\infty} |F(x_m) f(\tau(x_m))| \leq \lim_{m\to\infty} |F(x_m)| \max_{\overline{U_1}} |f| = 0$$

for all $f \in I_1[E]$. Therefore, since $T[E] = E$,

$$\lim_{m\to\infty} g(x_m) = 0$$

for all $g \in I_1[E]$. This obvious contradiction implies that $\tau^{-1}(U) \cap U = U$ and $F \neq 0$ on U.

Let $g \in E$ such that $T(g) = 1_U$. Let A be the set where $I_1(g) - I_1(g)1_U$ is nonzero. From property (1.8) and the fact that $\gamma(1_U)1_U = 1_U$, $\gamma(\chi_A)1_U = 0$ almost everywhere. However, since $S_1(I_1(g)) = 1_U$, $\gamma(\chi_A)1_{U'\setminus U} = 0$ almost everywhere. Therefore $|A| = 0$. Since $g \in E$, $A = \emptyset$ and $g = c1_U$. Because $\|1_U\|_p = \|c1_U\|_p$, $|c| = 1$. Moreover, by property (1.8), we have that $S_1(I_1(f)1_U) = cI_1(f)(\tau)1_U$ almost everywhere for all $f \in C(\overline{U})$. In addition, we have

$$\int_U |f(x)|^p \, dx = \int_U |f(\tau(x))|^p \, dx. \tag{3.11}$$

for all $f \in C(\overline{U})$. Since $S_1(I_1(x_i)) = I_1(\tau_i)$ each component of τ belongs to E. By Theorem 2 in [6], since τ is a mapping which is in $W_1^1(U)$ and the Jacobian J_τ satisfies $|J_\tau(x)|1_U = 1_U$, the change of variables in (3.11) holds for all nonnegative measureable functions, not just $|f(\cdot)|^p$ where $f \in C(\overline{U})$.

Since $S_1(I_1(x_i)) = I_1(\tau_i)$, property (1.8) implies

$$\partial_j \tau_i \partial_j \tau_m = 0 \tag{3.12}$$

almost everywhere for $i \neq m$. Hence, $\partial_i \tau_j \partial_m \tau_j = 0$ almost everywhere for $i \neq m$. Therefore, $J_\tau(x)$ is almost everywhere equal to an n by n invertible matrix with n nonzero entries.

From the use of Lamperti's Theorem, the mapping $f \mapsto f(\tau)$ is an $L^\infty(U)$ isometry as well. Therefore, since the real-valued functions of E are dense in the Banach space $C_{\mathbb{R}}(\overline{U})$ and $T[E] = [E]$, T has a surjective isometric extension mapping $C_{\mathbb{R}}(\overline{U})$ onto itself. Therefore, by the Banach-Stone Theorem, $\tau : \overline{U} \to \overline{U}$ must be a homeomorphism of $\overline{U}$ onto itself. Hence, $T^{-1}f = c^{-1}f(\tau^{-1})$ is a well defined isometry on $W_p^k(U)$ with the same properties as T.

Since E is dense in $L_p(U)$, (3.11) implies that T has an isometric extension mapping $L_p(U)$ onto itself. Therefore, if $k > 1$, we can interpolate between $L_p(U)$ and $W_p^k(U)$ to get that T has an isometric extension mapping $W_p^1(U)$ onto itself. Since $T(x_j) = c\tau_j$, it must be that τ_j is a linear function since

$$\|\tau_j\|_{k,p} = \|x_j\|_{k,p} = \|x_j\|_{1,p} = \|\tau_j\|_{1,p}.$$

By (3.12), $\tau_i = a_i x_{\sigma(i)} + b_i$ for some scalers a_i, b_i and a permutation σ of the set $\{1, \cdots, n\}$. Since $\|x_i\|_{1,p} = \|\tau_i\|_{1,p}$, $a_i = \pm 1$ by (3.11). Hence, τ is trivial.

For the remainder of this section, we assume $k = 1$. Since $T(x_j) = c\tau_j(x)$, it follows from (3.11) that

$$|U| = \sum_{i=1}^{n} \int_U |\partial_i \tau_j(x))|^p \ dx$$

Let $g_j \in C^\infty(\mathbb{R}^n)$ such that $g_j, \partial_j g_j > 0$ on $\overline{U}$ and $\partial_i g_j(x) = 0$ for all x and all $1 \leq i \neq j \leq n$. Since $g_j \in C^\infty$ and $\tau_i \in W_1^1(U)$ we can use the chain rule and (3.11) to see that

$$\int_U |\partial_j g_j(x)|^p \ dx = \int_U |\partial_j g_j(\tau(x))|^p \sum_{i=1}^{n} |\partial_j \tau_i(x)|^p \ dx.$$

Since $|\partial_j g_j(\tau(x))|^p \in L_1(U)$, $|\partial_j g_j(\tau(x))|^p > 0$ and U, equipped with Lebesgue measure, is a finite measure space, we can use the Hahn-Banach Theorem and the Riesz-Representation Theorem to prove that

$$\sum_{i=1}^{n} |\partial_j \tau_i(x)|^p \leq 1 \tag{3.13}$$

almost everywhere in U.

Let η_ϵ be a nonnegative, compactly supported, smooth approximate identity. Since $\tau_i 1_U \in E$, $u_m = \tau_i * \eta_{1/m}$ converges to $\tau_i 1_U$ in $W_p^k(U)$ and converges pointwise to $\tau_i 1_U$ in U. Moreover, we can obtain the following estimate for all $x, y \in U$.

$$\begin{aligned}
|u_m(x) - u_m(y)| &= \Big|\int_0^1 \tfrac{d}{dt} u_m(tx + (1-t)y) \ dt\Big| \\
&= \Big|\int_0^1 \nabla u_m(tx + (1-t)y) \cdot (x - y) \ dt\Big| \\
&\leq \sup_z |\nabla u_m(z)||x - y| \\
&= |x - y| \sup_z \Big(\sum_{j=1}^{n} \big(\partial_j \int_{\mathbb{R}^n} \tau_i(w)\eta_{1/m}(w - z) \ dw\big)^2\Big)^{1/2} \\
&= |x - y| \sup_z \Big(\sum_{j=1}^{n} \big(\int_{\mathbb{R}^n} \tau_i(w)\partial_j(\eta_{1/m}(w - z)) \ dw\big)^2\Big)^{1/2} \\
&= |x - y| \sup_z \Big(\sum_{j=1}^{n} \big(\int_{\mathbb{R}^n} \partial_j \tau_i(w)\eta_{1/m}(w - z) \ dw\big)^2\Big)^{1/2} \\
&\leq |x - y| \sup_z \sum_{j=1}^{n} \int_{\mathbb{R}^n} |\partial_j \tau_i(w)|\eta_{1/m}(w - z) \ dw, \qquad (\eta_{1/m} \geq 0)
\end{aligned}$$

$$\begin{aligned} &\leq \; |x-y| \sup_z \int_{\mathbb{R}^n} \eta_{1/m}(w-z)\, dw, \qquad &&\text{(3.12) and (3.13)} \\ &= \; |x-y| \end{aligned}$$

Since $u_m \to \tau_i$ pointwise in U, τ_i is Lipschitz with Lipschitz constant equal to 1. Hence, τ is a Lipschitz map with Lipschitz constant equal to 1. Since T is invertible, the same argument shows that τ^{-1} is also Lipschitz with Lipschitz constant equal to 1. Hence, τ is bi-Lipschitz with bi-Lipschitz constant equal to one.

Let $B \subset U$ be a ball. Then, $\tau : B \to \mathbb{R}^n$ is a bi-Lipschitz map with bi-Lipschitz constant equal to 1. By Theorem 2.8 in [2], B has the (C, δ)-*linear bilipschitz extension property.* This means that for a fixed $0 \leq \epsilon \leq \delta \leq 1$, each bi-Lipschitz map defined on B with bi-Lipschitz constant $1 + \epsilon$ can be extended to $\mathbb{R}^n$ by a bi-Lipschitz map with bi-Lipschitz constant $1 + \epsilon'$ where $\epsilon' = C\epsilon$. Since τ has bi-Lipschitz constant equal to 1, it can be extended to a bi-Lipschitz map with bi-Lipschitz constant equal to 1. Therefore, the bi-Lipschitz extension of $\tau : B \to \mathbb{R}^n$ is an isometry. Since U can be covered with a sequence of of balls B_m such that $B_m \cap B_{m+1} \neq \emptyset$, there is a unique isometry A of $\mathbb{R}^n$ for which $A\big|_U = \tau$. Since A is an isometry, it is affine. Therefore τ is affine and τ satisfies (3.12). So, $\tau_i(x) = a_i x_{\sigma(i)} + b_i$ for some constants a_i, b_i and a permutation σ of $\{1, \cdots, n\}$. By (3.11), $a_i \pm 1$. Therefore, τ is trivial and

$$T(f)(x) = cf(\tau(x))$$

for all $f \in W^k_p(U)$.

4. Lamperti's Theorem and the mapping τ, II

We will now consider the case $p \in 2\mathbb{N} \backslash \{2\}$. In this case, we apply Lamperti's Theorem to the operator S_2. This means that there exists an $F \in L_{p'}(U')$ and a regular set isomorphism ϕ, inducing a map γ such that

$$S_2(g)(x) = F(x)\gamma(g)(x)$$

almost everywhere for all $g \in L_{p'}(U')$.

Since all polynomials restricted to some component of U' are in $(W^k_p(U))^*$, define $\psi_i = S_2(x_i)$. By applying Lemma 1 and Lemma 2 to ψ, we have that $S_2(q)(x) = F(x)q(\psi(x))$ for all q which are linear combinations of functions $p_\alpha 1_{U_\alpha}$ where each p_α is a polynomial. Since such functions converge uniformly to functions in $C(\overline{U'})$, the same is true for all $f \in C(\overline{U'})$.

Since $T^*(L_f) = L_{T(f)}$ and $1_U \in C(\overline{U'})$, $S_2(I_2(L_{1_U})) = I_2(L_{T(1_U)})$ cannot vanish in U. Otherwise, since $I_2(L_{T(1_U)})1_U \in E$, there would be a point $x_0 \in U$ such that for all $g \in E$, $I_2(L_g)(x_0)1_U(x_0) = 0$. Moreover, by property (1.8), if $T(h) = 1_U$, then $I_2(h)$ is essentially supported in U because $h \in E$. Hence, $h = c1_U$ is a constant function and $S_2\big|_{L_{p'}(U)}$ is an isometry satisfying $S_2(f)(x) = cf(\psi(x))$ for all $f \in C(\overline{U})$.

Since $S_2(I_2(L_{T(f)})) = I_2(L_f)$, it follows that $cT(f)(\psi(x)) = f(x)$ and $T(f)(\psi) \in E$ whenever $f \in E$. Since $p > 2$ and U is bounded, we have that $W^k_p(U) \subset W^k_{p'}(U)$. Define an isometric embedding $I_3 : E \subset W^k_{p'}(U) \to L_{p'(U')}$ in the same way I_1 was defined. Then, the operator $\tilde{T} = I_3^{-1} S_2 I_3$ is a $W^k_{p'}(U)$ isometry mapping $E \subset W^k_{p'}(U)$ functions onto themselves. Moreover, we know that

$$\tilde{T}(f)(x) = cf(\psi(x))$$

for $f \in E$. Since U has a Lipschitz boundary, E is dense in $W^k_{p'}(U)$. Therefore, $\tilde{T}$ can be extended to a surjective isometry, also denoted by $\tilde{T}$, such that $\tilde{T}[E] = E$.

It should be noted that E may be a proper subset of $W^k_{p'}(U) \cap C(\overline{U})$. Therefore, we cannot finish the proof by noting that p' is not an even integer and applying the results of the previous section. However, the real valued functions of E are still dense in $C_{\mathbb{R}}(\overline{U})$. Therefore, by the Banach-Stone Theorem, ψ is a homeomorphism mapping $\overline{U}$ onto itself. So, $\tilde{T}$ and T have well defined inverses. Moreover, from Lamperties Theorem, (1.8) and (1.4), ψ satisfies all the same conditions of τ from the previous section. Therefore, ψ is trivial by the same reasoning. So, since $cT(f)(\psi(x)) = f(x)$, $T(f)(x) = c^{-1}f(\psi^{-1}(x))$ where $|c| = 1$ and ψ^{-1} is trivial.

$\square$

5. A SPECIAL THANKS

I would like to thank Ralph Howard for discussions regarding Sobolev mappings. Although the results these discussions do not appear in the proof of Theorem 1, they greatly improved my understanding of the general problem where continuous functions are not assumed to be invariant under the isometry. Also, I would like to thank Alexander Koldobsky for introducing me to this problem while I was a graduate student and writting part one of this article with me. Studying this problem has helped me understand much about the geometry of Banach spaces.

REFERENCES

[1] J. Adams and J. Fournier, *Sobolev Spaces,* Academic Press (2003).

[2] P. Alestalo, D.A. Trotsenko, and J. Väisälä, *Linear bilipschitz extension property,* Siberian Math. J., **44**, No. 6 (2003), 959-968.

[3] , G. Diestel and A. Koldobsky, *Sobolev spaces with only trivial isometries*, Positivity Vol.**10**, (2006), 135-144.

[4] R.J. Fleming and J.E. Jamison, *Isometries on Banach spaces: a survey*, In: Analysis, geometry and groups : a Riemann legacy volume 1, Hardronic Press, Palm Harbor, FL,(1993), 52-123.

[5] C.D. Hardin Jr., *Isometries on subspaces of L^p*, Indiana Univ. Math J. **30**(1981), 449-465.

[6] P. Hajlasz, *Change of variables formula under minimal assumptions,* Colloq. Math., **64** (1993), no. 1, 93-101.

[7] A. Koldobsky and H. König, *Aspects of Isometric Theory of Banach Spaces*, Handbook of the Geometry of Banach Spaces, Vol. **1**, Elsevier Science B.V.,(2001).

[8] J. Lamperti, *On the isometries of certain function spaces*, Pacific J.Math., **8**(1958),459-466.

[9] A. Plotkin, *Isometric operators in spaces of summable analytic and harmonic functions*, Soviet Mat. Dokl. **10** (1969), 461-463.

[10] A. Plotkin,*On isometric operators on subspaces of L^p*, Soviet Mat. Dokl. **11** (1970), 981-983.

[11] A. Plotkin, *Continuation of L^p isometries*, V.A. Steklova Akad. Nauk. SSSR **22** (1971), 103-129.

[12] W. Rudin, *L_p-isometries and equimeasurability*, Indiana Univ. Math J. **25**(1976), 215-228.

GEOFF DIESTEL, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC 29208, USA
E-mail address: `diestelg@math.sc.edu,koldobsk@math.missouri.edu`