

# SOME REMARKS ON BILINEAR LITTLEWOOD-PALEY THEORY

GEOFF DIESTEL

ABSTRACT. In this work, some bilinear analogues of linear Littlewood-Paley theory are explored. Paraproducts with functions decomposed at different scales are shown to be bounded on certain products of Lebesgue spaces. Results concerning square functions associated with smooth and nonsmooth cutoffs are also obtained.

## 1. INTRODUCTION

Interest in the study of bilinear operators has increased in light of the results of Lacey and Thiele [?],[?] on the boundedness of the bilinear Hilbert transform. They have proved that the characteristic function of a half-plane in  $\mathbf{R}^2$  is the symbol of a bounded bilinear multiplier on several products of Lebesgue spaces. Characteristic functions of other geometric sets have the same property. For instance, Grafakos and Li [?] have shown that the characteristic function of the unit disc in  $\mathbf{R}^2$  is the symbol of a bounded bilinear operator from  $L^p \times L^q$  to  $L^r$  whenever  $2 \leq p, q < \infty$ ,  $1 < r \leq 2$  and  $1/p + 1/q = 1/r$ . In this work, we study families of smooth and nonsmooth symbols of bounded bilinear operators arising in natural decompositions of the plane and we obtain the boundedness of square functions associated with them.

Using orthogonality considerations, one deduces that the sum of a sequence of uniformly bounded bilinear symbols on  $\mathbf{R}^2$  whose (compact) supports have pairwise disjoint projections on the  $\xi$ -axis,  $\eta$ -axis, and the line  $\eta = \xi$ , is also a bounded bilinear symbol on certain products of Lebesgue spaces, see [?]. In this article we are concerned with situations where a family of compactly supported symbols does not have this property.

The main purpose of our work is to study a bilinear approach to nonsmooth, multi-scale paraproducts. The standard paraproduct operator is of the form

$$T_g(f)(x) = \sum_j \Delta_j(g)(x) S_j(f)(x)$$

where  $f \in L_2$ ,  $g \in BMO$ ,  $\Delta_j$  is the standard Littlewood-Paley operator and

$$S_j = \sum_{i < j} \Delta_i.$$

The situation discussed in this paper is to consider  $a, b \in (0, 1)$  and the bilinear operator

$$S_{a,b}(f, g)(x) = \sum_j \Delta_j^a(f)(x) S_j^b(g)(x)$$

where  $f \in L_p$ ,  $g \in L_q$ ,  $\Delta_j^a$  is the nonsmooth Littlewood-Paley operator associated with the  $j$ -th lacunary scale of  $a$  and  $S_j^b = \sum_{i < j} \Delta_i^b$ . A nonsmooth version of the standard

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paraproduct would then be  $S_{1/2,1/2}$ . In this situation, the family of symbols consists of characteristic functions of disjoint rectangles

$$[a^j, a^{j-1}] \times [-b^j, b^j].$$

The sum of such symbols is shown to be bounded and, as a corollary, a square function associated with nested rectangular frames

$$([-a^{j-1}, a^{j-1}] \times [-b^j, b^j]) \setminus ([-a^j, a^j] \times [-b^{j+1}, b^{j+1}])$$

is also established to be bounded.

We also discuss vector-valued results as well as examples of smooth and nonsmooth decompositions of the plane. In the smooth case, the associated square function is shown to be bounded using results of Gilbert and Nahmod [?]. The nonsmooth case requires more delicate analysis and may not be treated using the techniques of this article. The main problem is that, unlike the linear case, one cannot simply derive nonsmooth Littlewood-Paley results from smooth ones.

## 2. ADJACENT BOXES WITH NESTED $\eta$ PROJECTIONS

Throughout this section we fix two parameters  $a, b \in (0, 1)$  and we let  $k$  run through the set of integers  $\mathbf{Z}$ . We consider the bilinear operator with symbol the characteristic function of the rectangle  $[a^k, a^{k-1}] \times [-b^k, b^k]$ , that is the singular integral operator

$$S_k(f, g)(x) = \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i(\xi + \eta)x} \chi_{[a^k, a^{k-1}]}(\xi) \chi_{[-b^k, b^k]}(\eta) d\xi d\eta.$$

The operator  $S_k$  is a product of a singular integral of  $f$  times a singular integral of  $g$  and it is an easy consequence of Hölder's inequality that it is bounded from  $L^p(\mathbf{R}) \times L^q(\mathbf{R})$  to  $L^r(\mathbf{R})$  whenever  $1/p + 1/q = 1/r$  and  $1 < p, q < \infty$ .

Our goal is to study the sum of the operators  $S_k$  and show that their sum is also bounded on products of Lebesgue spaces. One may not use simple orthogonality considerations to study this sum as the  $\eta$  projections of the symbols are nested. A more delicate analysis will be employed to carry out this task; in the heart of this analysis lies a crucial summation by parts argument inspired by the work of Thiele [?].

The sum of the  $S_k$ 's has symbol  $\sum_{k \in \mathbf{Z}} \chi_k$ , where  $\chi_k = \chi_{[a^k, a^{k-1}]}(\xi) \chi_{[-b^k, b^k]}(\eta)$ . It will be shown below that the operator with symbol  $\sum_{k \in \mathbf{Z}} \chi_k$  is bounded on certain products of Lebesgue spaces.

To obtain a sharper result we will fix a bounded sequence  $\gamma = \{\gamma_k\}$  of complex numbers which we will initially assume to be compactly supported, i.e.  $\gamma_k = 0$  for all but finitely many integers  $k$ . This minor assumption will be removed at the end but will become handy in several interchanges of sums and integrals.

Define a family of bilinear operators  $\{S_{a,b,\gamma}\}$  as follows:

$$S_{a,b,\gamma}(f, g)(x) = \sum_{k \in \mathbf{Z}} \gamma_k \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\xi) \widehat{A}_k(\xi) \widehat{g}(\eta) \widehat{B}_k(\eta) e^{2\pi i(\xi + \eta)x} d\xi d\eta,$$

where  $\widehat{A}_k(\xi) = \chi_{[a^k, a^{k-1}]}(\xi)$  and  $\widehat{B}_k(\eta) = \chi_{[-b^k, b^k]}(\eta)$ . We have the following theorem concerning  $S_{a,b,\gamma}$ .

**Theorem 1.** For each  $1 < p, q, r < \infty$  such that  $1/p + 1/q = 1/r$  and for any  $a, b \in (0, 1)$  there exists a constant  $C = C(a, b, p, q)$  such that for any compactly supported bounded sequence of complex numbers  $\gamma$  we have

$$\|S_{a,b,\gamma}(f, g)\|_r \leq C \|\gamma\|_{\ell^\infty} \|f\|_p \|g\|_q$$

for any  $f$  in  $L^p(\mathbf{R})$  and  $g$  in  $L^q(\mathbf{R})$ .

*Proof.* We first set some notation. Fix  $\gamma$  as above and set  $S = S_{a,b,\gamma}$ . Let  $S^-$  and  $S^+$  be restrictions of the sum in the definition of  $S$  to  $\mathbf{Z}^-$  and  $\mathbf{Z} \setminus \mathbf{Z}^-$  respectively. Let  $P = \frac{\log a}{\log b}$ ,  $M > \max\{1 + a^{-1}, b^{-1}\}$ , and  $N$  be a large positive integer such that

$$(2.1) \quad N > \max\left\{\frac{\log(M+1)}{-\log b} - 1, \frac{\log M}{-\log b} + 1, \frac{P \log(Mb-1)}{\log a} + P, \frac{\log(M+1)}{-\log a} - 1, \frac{\log(M+a^{-1})}{-\log b} + 1\right\}.$$

Note that if  $m(\xi, \eta)$  is the symbol of a bounded bilinear operator, then so is  $m(-\xi, -\eta)$ . Let  $\tilde{S}$  be the operator

$$\tilde{S}(f, g)(x) = \sum_{k \in \mathbf{Z}} \gamma_k \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\xi) \widehat{A}_k(\xi) \widehat{g}(\eta) \widehat{B}_k(\eta) e^{2\pi i(\xi+\eta)x} d\xi d\eta$$

where  $\widehat{A}_k(\xi) = \widehat{A}_k(-\xi)$  and  $\widehat{B}_k(\eta) = \widehat{B}_k(-\eta)$ . By symmetry,  $\widehat{B}_k(-\eta) = \widehat{B}_k(\eta)$ . Therefore,  $\widehat{B}_k(\eta) = \widehat{B}_k(\eta)$ . So, in order to show the boundedness of  $S$ , it will suffice to show the boundedness of  $\tilde{S}^+ + S^-$  when  $a \leq b$  and  $S^+ + \tilde{S}^-$  when  $b < a$ . The proofs for these two cases are essentially similar provided certain notational changes are made. Therefore, we will only consider the case when  $a \leq b$ .

For  $k \in \mathbf{Z}$  we define functions  $C_k$  on the Fourier transform side by setting:

$$(2.2) \quad \widehat{C}_k(\tau) = \begin{cases} \chi_{[-Mb^k, Mb^k]}(\tau), & k \geq 0 \\ \chi_{[-Ma^k, Ma^k]}(\tau), & k < 0. \end{cases}$$

This allows us to rewrite  $\tilde{S}^+$  as

$$\tilde{S}^+(f, g)(x) = \sum_{k \geq 0} \gamma_k \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\xi) \widehat{A}_k(\xi) \widehat{g}(\eta) \widehat{B}_k(\eta) \widehat{C}_k(\xi + \eta) e^{2\pi i(\xi+\eta)x} d\xi d\eta$$

because the algebraic sum of the supports of  $\widehat{A}_k$  and  $\widehat{B}_k$  is contained in the support of  $\widehat{C}_k$ . Similarly,

$$S^-(f, g)(x) = \sum_{k < 0} \gamma_k \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\xi) \widehat{A}_k(\xi) \widehat{g}(\eta) \widehat{B}_k(\eta) \widehat{C}_k(\xi + \eta) e^{2\pi i(\xi+\eta)x} d\xi d\eta.$$

Since  $\widehat{C}_k$  is real, it is easy to show that

$$\int_{\mathbf{R}} \widehat{C}_k(\xi + \eta) \overline{h(x)} e^{2\pi i(\xi+\eta)x} dx = \int_{\mathbf{R}} \overline{(C * h)(x)} e^{2\pi i(\xi+\eta)x} dx.$$

So, by pairing with another function  $h$  we can write  $\langle \tilde{S}^+(f, g), h \rangle$  as

$$(2.3) \quad I^+ = \sum_{k \geq 0} \gamma_k \int_{\mathbf{R}} (f * \tilde{A}_k)(x) (g * B_k)(x) \overline{(h * C_k)(x)} dx$$

and  $\langle S^-(f, g), h \rangle$  as

$$(2.4) \quad I^- = \sum_{k < 0} \gamma_k \int_{\mathbf{R}} (f * A_k)(x) (g * B_k)(x) \overline{(h * C_k)(x)} dx.$$

Let us start by establishing the boundedness for (??). The following algebraic identity will be used in the sequel: for numbers  $a_k, b_k, c_k$  we have

$$(2.5) \quad \sum_k a_k b_k c_k = \sum_k a_k \sum_{j=0}^m (b_{k+j} c_{k+j} - b_{k+j+1} c_{k+j+1}) + \sum_k a_k b_{k+m+1} c_{k+m+1}.$$

By (??),

$$N > \frac{\log(M+1)}{-\log b} - 1.$$

From this it follows that

$$-a^k + b^{Pk+N+1} < -Mb^{Pk+N+1}.$$

This means that the algebraic sum of the supports of  $\widehat{A}_k$  and  $\widehat{B_{Pk+N+1}}$  lies to the left of the support of  $\widehat{C_{Pk+N+1}}$ . So, using (??), we can write (??) as

$$(2.6) \quad \sum_{k \geq 0} \gamma_k \int_{\mathbf{R}} (f * \widetilde{A}_k)(x) \sum_{j=0}^{(P-1)k+N} \left[ (g * B_{k+j})(x) \overline{(h * C_{k+j})(x)} \right. \\ \left. - (g * B_{k+j+1})(x) \overline{(h * C_{k+j+1})(x)} \right] dx \\ + \sum_{k \geq 0} \gamma_k \int_{\mathbf{R}} (f * \widetilde{A}_k)(x) (g * B_{Pk+N+1})(x) \overline{(h * C_{Pk+N+1})(x)} dx,$$

where the second sum in (??) is identically zero.

Rearranging the terms, the first sum in (??) is equal to

$$(2.7) \quad \sum_{k > N} \int_{\mathbf{R}} \sum_{j=0}^{\frac{(P-1)k+N}{P}} \gamma_{k-j} (f * \widetilde{A}_{k-j})(x) \left[ (g * B_k)(x) \overline{(h * C_k)(x)} \right. \\ \left. - (g * B_{k+1})(x) \overline{(h * C_{k+1})(x)} \right] dx$$

$$(2.8) \quad + \sum_{k=1}^N \int_{\mathbf{R}} \sum_{j=0}^{k-1} \gamma_{k-j} (f * \widetilde{A}_{k-j})(x) \left[ (g * B_k)(x) \overline{(h * C_k)(x)} \right. \\ \left. - (g * B_{k+1})(x) \overline{(h * C_{k+1})(x)} \right] dx.$$

For  $s < t$ , we have that

$$\left\| \sup_t \sup_{s < t} |(\widehat{f}\chi_{[s,t]})^\vee| \right\|_p \leq \left\| \sup_t |(\widehat{f}\chi_{[-\infty,t]})^\vee| \right\|_p + \left\| \sup_s |(\widehat{f}\chi_{[-\infty,s]})^\vee| \right\|_p$$

The Carleson-Hunt theorem [?], [?] says that for all  $1 < p < \infty$  there is a constant  $C(p)$  such that for all functions  $f \in L^p(\mathbf{R})$  we have

$$(2.9) \quad \left\| \sup_t |(\widehat{f}\chi_{[-\infty,t]})^\vee| \right\|_p \leq C(p) \|f\|_p$$

Therefore (??) consists of only finitely many terms, each of which can be controlled by Hölder's Inequality and (??) as follows:

$$\begin{aligned} & \left| \int_{\mathbf{R}} (f * \tilde{A}_{k-j})(x)(g * B_k)(x) \overline{(h * C_k)(x)} \right| \\ & \leq C(p, q) \|\gamma\|_{L^\infty} \|f * \tilde{A}_{k-j}\|_p \|g * B_k\|_q \|h * C_k\|_{r'} \\ & \leq C(p, q) \|\gamma\|_{L^\infty} \|f\|_p \|g\|_q \|h\|_{r'}. \end{aligned}$$

On the other hand, (??) can be rewritten as  $I_1 + I_2 + I_3$  where

$$(2.10) \quad I_1 = \sum_{k>N} \int_{\mathbf{R}} \sum_{j=0}^{\frac{(P-1)k+N}{P}} \gamma_{k-j} (f * \tilde{A}_{k-j})(x)(g * B_k)(x) \left[ \overline{h * (C_k - C_{k+1})(x)} \right] dx,$$

$$(2.11) \quad I_2 = \sum_{k>N} \int_{\mathbf{R}} \sum_{j=0}^{\frac{(P-1)k+N}{P}} \gamma_{k-j} (f * \tilde{A}_{k-j})(x) \left[ (g * (B_k - B_{k+1}))(x) \right] \left[ \overline{h * (C_{k+1} - C_{k(i)+N})(x)} \right] dx,$$

$$(2.12) \quad I_3 = \sum_{k>N} \int_{\mathbf{R}} \sum_{j=0}^{\frac{(P-1)k+N}{P}} \gamma_{k-j} (f * \tilde{A}_{k-j})(x) \left[ g * (B_k - B_{k+1})(x) \right] \overline{(h * C_{k+N})(x)} dx.$$

Let us begin by bounding (??), which we can write as  $I_{11} + I_{12}$  where

$$(2.13) \quad I_{11} = \sum_{k>N} \sum_{j=0}^{\frac{(P-1)k-N}{P}} \gamma_{k-j} \int_{\mathbf{R}} (f * \tilde{A}_{k-j})(x)(g * B_k)(x) \left[ \overline{h * (C_k - C_{k+1})(x)} \right] dx$$

and

$$(2.14) \quad I_{12} = \sum_{k>N} \sum_{j=\frac{(P-1)k-N}{P}}^{\frac{(P-1)k+N}{P}} \gamma_{k-j} \int_{\mathbf{R}} (f * \tilde{A}_{k-j})(x)(g * B_k)(x) \left[ \overline{h * (C_k - C_{k+1})(x)} \right] dx.$$

By (??),

$$N > \frac{P \log(Mb-1)}{\log a} + P,$$

we have

$$-Mb^{k+1} < -a^{\frac{k+N}{P}-1} - b^k.$$

It is also easy to see that

$$b^k < Mb^{k+1}$$

because  $M > b^{-1}$ . Therefore,

$$-a^k + b^k < Mb^{k+1}$$

and the algebraic sum of the supports of  $\widehat{A}_{k-j}$  and  $\widehat{B}_k$  lies between the two disjoint segments of the support of  $\widehat{C}_k - \widehat{C}_{k+1}$  for all  $0 \leq j \leq \frac{(P-1)k-N}{P}$ . This proves that (??) is zero.

The term (??) is equal to at most  $2N + 1$  terms of the form

$$\sum_{k>N} \gamma_{k-k'} \int_{\mathbf{R}} (f * \tilde{A}_{k-k'})(x) (g * B_k)(x) \left[ \overline{h * (C_k - C_{k+1})(x)} \right] dx,$$

where  $k'$  is in the interval  $[\frac{(P-1)k-N}{P}, \frac{(P-1)k+N}{P}]$ . For  $R > \frac{2N}{P-1}$ ,  $\frac{(P-1)k+N}{P} < \frac{(P-1)(k+R)-N}{P}$ .

This means that  $\widehat{\tilde{A}_{k-k'}}$  and  $\widehat{\tilde{A}_{(k+R)-(k+R)'}}$  have disjoint supports. Therefore, (??) is equal to at most  $R(2N + 1)$  terms that can be controlled as follows:

$$\begin{aligned} &\leq \int_{\mathbf{R}} \|\gamma\|_{\infty} \left( \sum_{k>N} \left| (f * \tilde{A}_{k-k'})(x) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{k>N} \left| h * (C_k - C_{k+1})(x) \right|^2 \right)^{\frac{1}{2}} \sup_k \left| (g * B_k)(x) \right| dx \\ &\leq \|\gamma\|_{\infty} \int_{\mathbf{R}} \left( \sum_{k \geq 0} \left| (f * \tilde{A}_k)(x) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq 0} \left| h * (C_k - C_{k+1})(x) \right|^2 \right)^{\frac{1}{2}} \sup_{k \in \mathbf{Z}^+} \left| (g * B_k)(x) \right| dx \\ &\leq C(p, q) \|\gamma\|_{\infty} \|f\|_p \|g\|_q \|h\|_{r'} \end{aligned}$$

which follows from the Littlewood-Paley theorem and the fact that  $\sup_k |g * B_k|$  is controlled by (??).

In order to bound (??), notice that

$$\sup_k \left| \frac{(P-1)k+N}{P} \sum_{j=0}^k (f * \tilde{A}_{k-j}) \right| = \sup_k \left| \sum_{j=\frac{k-N}{P}}^k (f * \tilde{A}_j) \right|$$

For each  $k$ , let  $\ell(k) = \min_{\frac{k-N}{P} \leq j \leq k} \{-a^{j-1}\}$  and  $r(k) = \max_{\frac{k-N}{P} \leq j \leq k} \{-a^j\}$ . We now have

$$\sup_k \left| \sum_{j=\frac{k-N}{P}}^k (f * \tilde{A}_j) \right| = \sup_k \left| \left( \sum_{j=1}^{\infty} \widehat{(f * \tilde{A}_j) \chi_{[\ell(k), r(k)]}} \right)^{\vee} \right|.$$

By (??),

$$\left\| \sup_k \left| \left( \sum_{j=1}^{\infty} \widehat{(f * \tilde{A}_j) \chi_{[\ell(k), r(k)]}} \right)^{\vee} \right| \right\|_p \leq C(p) \left\| \sum_{j=1}^{\infty} (f * \tilde{A}_j) \right\|_p.$$

Then using the Marcinkiewicz multiplier theorem we conclude,

$$\left\| \sum_{j=1}^{\infty} (f * \tilde{A}_j) \right\|_p \leq C(p) \|f\|_p.$$

Now, since the supports of  $\widehat{B_k} - \widehat{B_{k+1}}$  and  $\widehat{C_{k+1}} - \widehat{C_{k+N}}$  are lacunary, the Littlewood-Paley theorem implies that

$$\left\| \left( \sum_{k>N} \left| g * (B_k - B_{k+1}) \right|^2 \right)^{1/2} \right\|_q \leq C(q) \|g\|_q$$

and

$$\left\| \left( \sum_{k>N} |h * (C_{k+1} - C_{k+N})|^2 \right)^{1/2} \right\|_{r'} \leq C(r') \|h\|_{r'}.$$

Therefore,

$$I_2 \leq C(p, q) \|\gamma\|_\infty \|f\|_p \|g\|_q \|h\|_{r'}.$$

To bound (??), let us notice that  $\text{supp}(\widehat{B}_k - \widehat{B}_{k+1}) = [-b^k, -b^{k+1}] \cup [b^{k+1}, b^k]$ . Let us write (??) as  $I_3^L + I_3^R$ , where  $L$  and  $R$  refer to the left and right halves of the

$$\text{supp}(\widehat{B}_k - \widehat{B}_{k+1}).$$

By (??),

$$N > \frac{\log M}{-\log b} + 1.$$

Therefore,  $b^{-N+1} > M$ . So, for all  $k \geq 0$  we have

$$\frac{a^k}{b^{k+N}} + b^{-N+1} > M$$

and therefore

$$-a^k - b^{k+1} < -Mb^{k+N}.$$

This means that  $I_3^L$  is identically zero because the algebraic sum of the support of  $\widehat{A}_{k-j}$  and  $[-b^k, -b^{k+1}]$  lies strictly to the left of the support of  $\widehat{C}_{k+N}$  for all

$$0 \leq j \leq \frac{(P-1)k+N}{P}.$$

Moreover, since

$$N > \frac{\log(M+a^{-1})}{-\log b} + 1,$$

$$Mb^{k+N} < -a^{\frac{k+N}{P}-1} + b^{k+1}.$$

From this,  $I_3^R$  can be reduced to

$$\sum_{k>N} \int_{\mathbf{R}} \sum_{\frac{(P-1)k-N}{P} < j \leq \frac{(P-1)k+N}{P}} \gamma_{k-j} (f * \widetilde{A}_{k-j})(x) (g * (B_k - B_{k+1}))(x) \overline{(h * C_{k+N})(x)} dx$$

because for  $0 \leq j < \frac{(P-1)k-N}{P}$ , the algebraic sum of the support of  $\widehat{A}_{k-j}$  and  $[b^{k+1}, b^k]$  lies strictly to the right of the support of  $\widehat{C}_{k+N}$ . The above can now be controlled in a similar way to how (??) was controlled. This time control  $\sup_k |(h * C_{k+N})|$  by (??).

Now, let us establish the boundedness of (??).

$$\begin{aligned} I^- &= \sum_{k=-1}^{-N} \gamma_k \int_{\mathbf{R}} (f * A_k)(x) (g * B_k)(x) \overline{(h * C_k)(x)} dx \\ &\quad + \sum_{k < -N} \gamma_k \int_{\mathbf{R}} (f * A_k)(x) (g * B_k)(x) \overline{(h * C_k)(x)} dx \end{aligned}$$

As we have seen already, the first sum in the previous equation is controlled by (??), while the second can be rewritten as

$$\begin{aligned} & \sum_{k < -N} \int_{\mathbf{R}} \gamma_k(f * A_k)(x) \sum_{j=0}^N \left[ (g * B_{k+j})(x) \overline{(h * C_{k+j})(x)} \right. \\ & \quad \left. - (g * B_{k+j+1})(x) \overline{(h * C_{k+j+1})(x)} \right] dx \\ & + \sum_{k < -N} \int_{\mathbf{R}} \gamma_k(f * A_k)(x) (g * B_{k+N+1})(x) \overline{(h * C_{k+N+1})(x)} dx. \end{aligned}$$

using (??).

By (??),

$$N > \frac{\log(M+1)}{-\log a} - 1.$$

So, for all  $k < 0$ ,

$$Ma^{k+N+1} < a^k - a^{k+N+1}.$$

Since  $k < -N$ ,  $-a^{k+N+1} < -b^{k+N+1}$ . Therefore, the second sum above is zero because

$$Ma^{k+N+1} < a^k - b^{k+N+1}$$

and the algebraic sum of the supports of  $\widehat{A}_k$  and  $\widehat{B}_{k+N+1}$  lies to the right of the support of  $\widehat{C}_{k+N+1}$ . The first sum can now be rewritten as:

$$\begin{aligned} & \sum_{k < -N} \int_{\mathbf{R}} \sum_{j=0}^N \gamma_{k-j}(f * A_{k-j})(x) (g * B_k)(x) \left[ \overline{(h * C_k)(x)} - \overline{(h * C_{k+1})(x)} \right] dx \\ & + \sum_{k < -N} \int_{\mathbf{R}} \sum_{j=0}^N \gamma_{k-j}(f * A_{k-j})(x) \left[ (g * B_k)(x) - (g * B_{k+1})(x) \right] \overline{(h * C_{k+1})(x)} dx. \end{aligned}$$

This in turn is at most  $N + 1$  terms of the form

$$\begin{aligned} & \sum_{k < -N} \int_{\mathbf{R}} \gamma_{k-k'}(f * A_{k-k'})(x) (g * B_k)(x) \left[ \overline{(h * C_k)(x)} - \overline{(h * C_{k+1})(x)} \right] dx \\ & + \sum_{k < -N} \int_{\mathbf{R}} \gamma_{k-k'}(f * A_{k-k'})(x) \left[ (g * B_k)(x) - (g * B_{k+1})(x) \right] \overline{(h * C_{k+1})(x)} dx \end{aligned}$$

where  $k'$  is an integer in the interval  $[0, N]$ . As we have seen before, this is bounded by a constant multiple of

$$\begin{aligned} & \|\gamma\|_{\infty} \int_{\mathbf{R}} \left( \sum_{k < -N} \left| (f * A_{k-k'})(x) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{k < -N} \left| (h * C_k)(x) - (h * C_{k+1})(x) \right|^2 \right)^{\frac{1}{2}} \\ & \quad \sup_k \left| (g * B_k)(x) \right| dx \\ & + \|\gamma\|_{\infty} \int_{\mathbf{R}} \int_{\mathbf{R}} \left( \sum_{k < -N} \left| (f * A_{k-k'})(x) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{k < -N} \left| (g * B_k)(x) - (g * B_{k+1})(x) \right|^2 \right)^{\frac{1}{2}} \\ & \quad \sup_k \left| (h * C_{k+1})(x) \right| dx \end{aligned}$$

where both of these sums can be controlled by a constant multiple of

$$\|f\|_p \|g\|_q \|h\|_{r'}.$$

A similar proof establishes Theorem ?? for  $b < a$ . Therefore, Theorem ?? holds for all compactly supported and bounded sequences  $\gamma$ . By a simple limiting argument, this result extends to arbitrary bounded sequences  $\gamma$ .  $\square$

Now, let  $\gamma = \{\gamma_k\}$  be a bounded sequence and let  $a, b \in (0, 1)$ . Consider the bilinear operators

$$S_k(f, g)(x) = \gamma_k \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(\xi) \widehat{A}_k(\xi) \widehat{g}(\eta) \widehat{B}_k(\eta) e^{2\pi i(\xi+\eta)x} d\xi d\eta,$$

where  $\widehat{A}_k(\xi) = \chi_{[a^k, a^{k-1}]}(\xi)$  and  $\widehat{B}_k(\eta) = \chi_{[-b^k, b^k]}(\eta)$ . By Theorem 1, the sum of these operators is bounded from  $L^p(\mathbf{R}) \times L^q(\mathbf{R})$  to  $L^r(\mathbf{R})$  for  $1/p + 1/q = 1/r$  and  $1 < p, q, r < \infty$ . As one might expect, the square function associated with these operators is also bounded.

**Corollary 1.** *For all  $1 < p, q, r < \infty$  such that  $1/p + 1/q = 1/r$  there exists a constant  $C = C(a, b, p, q)$  such that for all functions  $f \in L^p(\mathbf{R})$  and  $g \in L^q(\mathbf{R})$  the following estimate holds.*

$$\left\| \left( \sum_{k \in \mathbf{Z}} |S_k(f, g)|^2 \right)^{1/2} \right\|_r \leq C \|\gamma\|_\infty \|f\|_p \|g\|_q$$

*Proof.* We use Khinchine's Inequality to linearize the problem. Let  $r_k$  denote the Rademacher functions. Then we have

$$\left\| \left( \sum_{k \in \mathbf{Z}} |S_k(f, g)|^2 \right)^{1/2} \right\|_r \leq C(r) \left( \int_0^1 \left\{ \int_{\mathbf{R}} \left| \sum_{k \in \mathbf{Z}} S_k(f, g)(x) r_k(w) \right|^r dx \right\} dw \right)^{1/r}$$

using Fubini's theorem. The expression inside the curly brackets was shown in Theorem 1 to be bounded by a constant multiple of  $\|f\|_p^r \|g\|_q^r$  uniformly in the choice of  $r_k(w)$ . The required conclusion follows.  $\square$

Now, consider the bilinear operators  $S_k$  with symbols  $m_k(\xi, \eta)$ , where

$$m_k(\xi, \eta) = \gamma_k (\chi_{[-a^{k-1}, a^{k-1}]}(\xi) \chi_{[-b^{k-1}, b^{k-1}]}(\eta) - \chi_{[-a^k, a^k]}(\xi) \chi_{[-b^k, b^k]}(\eta))$$

**Corollary 2.** *For all  $1 < p, q, r < \infty$  such that  $1/p + 1/q = 1/r$  there exists a constant  $C = C(a, b, p, q)$  such that for all functions  $f \in L^p(\mathbf{R})$  and  $g \in L^q(\mathbf{R})$  the following estimate holds*

$$\left\| \left( \sum_{k \in \mathbf{Z}} |S_k(f, g)|^2 \right)^{1/2} \right\|_r \leq C \|\gamma\|_\infty \|f\|_p \|g\|_q.$$

*Proof.* We will linearize the problem using Rademacher functions and Fubini's theorem as in the previous corollary.

$$\left\| \left( \sum_{k \in \mathbf{Z}} |S_k(f, g)|^2 \right)^{1/2} \right\|_r \leq C(r) \left( \int_0^1 \left\{ \int_{\mathbf{R}} \left| \sum_{k \in \mathbf{Z}} S_k(f, g)(x) r_k(w) \right|^r dx \right\} dw \right)^{1/r}$$

For each  $k \in \mathbf{Z}$  write  $S_k = \sum_{i=1}^8 S_k^{(i)}$  where each  $S_k^{(i)}$  has symbol  $m_k^{(i)}$  defined by

$$m_k^{(1)}(\xi, \eta) = \chi_{[a^k, a^{k-1}]}(\xi) \chi_{[-b^k, b^k]}(\eta), \quad m_k^{(2)}(\xi, \eta) = \chi_{[a^k, a^{k-1}]}(\xi) \chi_{[b^k, b^{k-1}]}(\eta),$$

$$m_k^{(3)}(\xi, \eta) = \chi_{[-a^k, a^k]}(\xi) \chi_{[b^k, b^{k-1}]}(\eta), \quad m_k^{(4)}(\xi, \eta) = \chi_{[-a^{k-1}, -a^k]}(\xi) \chi_{[b^k, b^{k-1}]}(\eta),$$

$$m_k^{(5)}(\xi, \eta) = \chi_{[-a^{k-1}, -a^k]}(\xi) \chi_{[-b^k, b^k]}(\eta), \quad m_k^{(6)}(\xi, \eta) = \chi_{[-a^{k-1}, -a^k]}(\xi) \chi_{[-b^{k-1}, -b^k]}(\eta),$$

$$m_k^{(7)}(\xi, \eta) = \chi_{[-a^k, a^k]}(\xi) \chi_{[-b^{k-1}, -b^k]}(\eta), \quad m_k^{(8)}(\xi, \eta) = \chi_{[a^k, a^{k-1}]}(\xi) \chi_{[-b^{k-1}, -b^k]}(\eta)$$

For  $i = 1, 3, 5, 7$ , Theorem 1 implies that

$$\left\| \sum_{k \in \mathbf{Z}} S_k^{(i)}(f, g) \right\|_r \leq C(a, b, p, q) \|\gamma\|_\infty \|f\|_p \|g\|_q.$$

The remaining cases  $i = 2, 4, 6, 8$  are symmetric and it suffices to consider say the term  $i = 2$ . We introduce nonsmooth Littlewood-Paley operators  $\Delta_k^a(f) = (\widehat{f} \chi_{[a^k, a^{k-1}]})^\vee$  and  $\Delta_k^b(g) = (\widehat{g} \chi_{[b^k, b^{k-1}]})^\vee$ . Then we use the Cauchy-Schwarz and Hölder's inequalities to deduce

$$\begin{aligned} \left\| \sum_{k \in \mathbf{Z}} S_k^{(2)}(f, g) \right\|_r &= \left( \int_{\mathbf{R}} \left| \sum_{k \in \mathbf{Z}} \gamma_k \Delta_k^a(f)(x) \Delta_k^b(g)(x) \right|^r dx \right)^{\frac{1}{r}} \\ &\leq \|\gamma\|_\infty \left( \int_{\mathbf{R}} \left| \left( \sum_{k \in \mathbf{Z}} |\Delta_k^a(f)(x)|^2 \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbf{Z}} |\Delta_k^b(g)(x)|^2 \right)^{\frac{1}{2}} \right|^r dx \right)^{\frac{1}{r}} \\ &\leq C(p, q) \|\gamma\|_\infty \left\| \left( \sum_{k \in \mathbf{Z}} |\Delta_k^a(f)|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_{k \in \mathbf{Z}} |\Delta_k^b(g)|^2 \right)^{1/2} \right\|_q \\ &\leq C'(p, q) \|\gamma\|_\infty \|f\|_p \|g\|_q. \end{aligned}$$

□

### 3. SMOOTH AND NONSMOOTH DECOMPOSITIONS OF THE PLANE

One may consider bilinear operators of the form

$$(3.15) \quad \sum_j \sum_k a_{j,k} T_{j,k}(f, g),$$

where  $T_{j,k}$  has symbol  $m_{j,k}(\xi, \eta) = \chi_{I_j}(\xi) \chi_{I'_k}(\eta)$ , for two sequences of pairwise disjoint intervals  $\{I_j\}$  and  $\{I'_k\}$  and a bounded sequence of numbers  $a_{j,k}$  that satisfies  $|a_{j,k}| \leq 1$ . It is a natural question to ask whether the operator in (??) is bounded from  $L^p \times L^q$  into  $L^r$

for  $1/p + 1/q = 1/r$ . This may not be the case for all sequences  $\{a_{j,k}\}$ . Take for example  $a_{j,k} = b_j c_k$ , where  $|b_j|, |c_k| \leq 1$ . Then the operator in (??) reduces to the product

$$(3.16) \quad \left[ \sum_j b_j (\widehat{f}\chi_{I_j})^\vee \right] \left[ \sum_k c_k (\widehat{g}\chi_{I'_k})^\vee \right]$$

and one may easily see that either term in (??) can be unbounded for some sequences  $\{b_j\}$  and  $\{c_k\}$  on  $L^p(\mathbf{R})$ , whenever  $1 < p < \infty$  and  $p \neq 2$ . For instance, taking  $b_k = \pm 1$ ,

$$f \mapsto \sum_j \pm (\widehat{f}\chi_{I_j})^\vee$$

cannot be bounded on  $L^p(\mathbf{R})$ , since otherwise it would be bounded on  $L^{p'}(\mathbf{R})$  as well. By using Khinchine's Inequality, one could then show that

$$(3.17) \quad \text{Avg}_\epsilon \left\| \sum_j \epsilon_j (\widehat{f}\chi_{I_j})^\vee \right\|_s \approx \left\| \left( \sum_j |(\widehat{f}\chi_{I_j})^\vee|^2 \right)^{1/2} \right\|_s$$

for  $s = \min\{p, p'\}$ . However, we know that (??) can only hold when  $s \geq 2$ , see [?] (and also [?]) for counterexamples.

However, if we consider a bilinear square function instead of (??), we have a simple proposition below that makes use of the Rubio de Francia [?] square function theorem.

**Proposition 1.** *Let  $\{I_j\}_j$  and  $\{J_k\}_k$  be two sequences of disjoint intervals in  $\mathbf{R}$ . Let  $\{T_{j,k}\}$  be a family of bounded bilinear operators from  $L^p(\mathbf{R}) \times L^q(\mathbf{R})$  into  $L^r(\mathbf{R})$  with symbols  $m_{j,k}(\xi, \eta) = \chi_{I_j}(\xi)\chi_{J_k}(\eta)$ . Suppose  $2 \leq p, q < \infty$  and  $1/p + 1/q = 1/r$ . Then there is a constant  $C = C(p, q)$  such that for all  $f, g$*

$$\left\| \left( \sum_j \sum_k |T_{j,k}(f, g)|^2 \right)^{1/2} \right\|_r \leq C \|f\|_p \|g\|_q$$

*Proof.*

$$\begin{aligned} \left\| \left( \sum_j \sum_k |T_{j,k}(f, g)|^2 \right)^{1/2} \right\|_r &= \left\| \left( \sum_j \sum_k |(\widehat{f}\chi_{I_j})^\vee (\widehat{g}\chi_{J_k})^\vee|^2 \right)^{1/2} \right\|_r \\ &\leq \left\| \left( \sum_j |(\widehat{f}\chi_{I_j})^\vee|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_k |(\widehat{g}\chi_{J_k})^\vee|^2 \right)^{1/2} \right\|_q \\ &\leq C \|f\|_p \|g\|_q \quad (\text{using [?].}) \end{aligned}$$

□

We note that if the endpoints of the intervals in Proposition 1 form lacunary sequences, then the Littlewood-Paley theorem can be used instead of [?] and in this case the indices  $p, q$  can be taken in the range  $1 < p, q < \infty$ .

Next, we discuss two smooth dyadic decompositions in which the associated square functions can be shown to be bounded using results of Gilbert and Nahmod [?]. The nonsmooth analogs of these decompositions require more delicate analysis which is still elusive at this point. Unlike the linear Littlewood-Paley theory, nonsmooth results are not simple corollaries of smooth ones.

The first example we look at is the decomposition of certain half-planes by dyadic infinite strips. The second is a decomposition of a cone into dyadic parallelograms. The following theorems can be found in [?]. Define the half-planes  $P_\theta$  as follows.

$$P_\theta = \left\{ (\xi, \eta) : \xi \tan \theta - \eta > 0 \right\}.$$

**Theorem A.** *Let  $m(\xi, \eta)$  be a function having derivatives of all orders in the half-plane  $P_\theta$  such that*

$$|D^\alpha m(\xi, \eta)| \leq C \left( \text{dist}((\xi, \eta), \partial P_\theta) \right)^{-|\alpha|}, \quad |\alpha| \geq 0.$$

*Let  $T$  be the bilinear operator with symbol  $m$ . Then if  $\partial P_\theta$  is not one of the coordinate axes,  $T$  is bounded from  $L^p(\mathbf{R}) \times L^q(\mathbf{R})$  into  $L^r(\mathbf{R})$  for  $1/p + 1/q = 1/r < 3/2$ .*

**Theorem B.** *Let  $\Gamma$  be a closed one-sided cone with vertex at the origin and  $m(\xi, \eta)$  a function having derivatives of all orders inside  $\Gamma$  such that*

$$|D^\alpha m(\xi, \eta)| \leq C \left( \text{dist}((\xi, \eta), \partial P_\theta) \right)^{-|\alpha|}, \quad |\alpha| \geq 0.$$

*Then the bilinear operator  $T$  with symbol  $m$  is bounded from  $L^p(\mathbf{R}) \times L^q(\mathbf{R})$  into  $L^r(\mathbf{R})$  for  $1/p + 1/q = 1/r < 3/2$ , so long as no edge of  $\Gamma$  lies on the diagonal  $\xi - \eta = 0$  or a coordinate axis.*

Consider a Schwartz function  $\phi$  such that  $\chi_{(1,2)} \leq \phi \leq \chi_{(1/2,4)}$ . For  $\theta \in (0, \pi/2)$ , define a family of symbols  $m_k$  by:

$$m_k(\xi, \eta) = \phi \left( 2^{-k} \left( \xi - \frac{\eta}{\tan \theta} \right) \right).$$

Let  $T_k$  be the bilinear operator associated with the symbol  $m_k$ . Also, for  $\theta_1, \theta_2 \in (0, \pi/4)$  such that  $\theta_1 + \theta_2 \neq \pi/4$  define  $m_{j,k}$  as follows:

$$m_{j,k}(\xi, \eta) = \phi \left( 2^{-j} (\eta - \xi \tan \theta_1) \right) \phi \left( 2^{-k} \left( \xi - \frac{\eta}{\tan \theta_2} \right) \right).$$

Let  $T_{j,k}$  be the operator with symbol  $m_{j,k}$ .

**Corollary 3.** *For  $1/p + 1/q = 1/r < 3/2$ , there exists a constant  $C < \infty$  such that*

$$\left\| \left( \sum_{k \in \mathbf{Z}} |T_k(f, g)|^2 \right)^{1/2} \right\|_r \leq C \|f\|_p \|g\|_q.$$

*Proof.* First, we linearize the problem using the Rademacher functions and Fubini's theorem:

$$\left\| \left( \sum_{k \in \mathbf{Z}} |T_k(f, g)|^2 \right)^{1/2} \right\|_r \leq C(r) \left( \int_0^1 \int_{\mathbf{R}} \left| \sum_{k \in \mathbf{Z}} T_k(f, g)(x) r_k(w) \right|^r dx dw \right)^{\frac{1}{r}}.$$

Let  $R_w(f, g) = \sum_{k \in \mathbf{Z}} T_k(f, g) r_k(w)$ .  $R_w$  has symbol  $\sum_{k \in \mathbf{Z}} m_k(\xi, \eta) r_k(w)$  which is  $C^\infty$  in the half-plane. Fix  $(\xi, \eta)$ . Notice that  $(\xi, \eta)$  is in the support of at most three of the  $m_k$ 's. Therefore, there exists a  $j \in \mathbf{Z}$  such that

$$\left| D^\alpha \sum_{k \in \mathbf{Z}} m_k(\xi, \eta) r_k(w) \right| \leq C \left| D^\alpha \left( \phi \left( 2^{-j} \left( \xi - \frac{\eta}{\tan \theta} \right) \right) \right) \right| \leq C(\alpha, \theta) 2^{-j|\alpha|}.$$

By construction,

$$\text{dist}((\xi, \eta), \partial P_\theta) \leq C 2^j.$$

Now applying Theorem A we conclude that

$$\begin{aligned} C(r) \left( \int_0^1 \int_{\mathbf{R}} |T_k(f, g)(x) r_k(w)|^r dx dw \right)^{\frac{1}{r}} &\leq C(r) \left( \int_0^1 \left\| \sum_{k \in \mathbf{Z}} T_k(f, g) r_k(w) \right\|_r^r dw \right)^{\frac{1}{r}} \\ &\leq C(r, \theta) \|f\|_p \|g\|_q \end{aligned}$$

and this concludes the proof of the corollary.  $\square$

**Corollary 4.** *For  $1/p + 1/q = 1/r < 3/2$ , there exists a constant  $C < \infty$  such that*

$$\left\| \left( \sum_{k \in \mathbf{Z}} |T_{j,k}(f, g)|^2 \right)^{1/2} \right\|_r \leq C \|f\|_p \|g\|_q.$$

*Proof.* In this case, linearizing using Rademacher functions and Fubini's theorem is done as can be seen in [?].

$$\left\| \left( \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} |T_{j,k}(f, g)|^2 \right)^{1/2} \right\|_r \leq C_r \left( \int_0^1 \int_0^1 \int_{\mathbf{R}} \left| \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} T_{j,k}(f, g)(x) r_j(w) r_k(z) \right|^r dx dw dz \right)^{\frac{1}{r}}.$$

Let

$$R_{w,z}(f, g) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} T_{j,k}(f, g) r_j(w) r_k(z).$$

$R_{w,z}$  has symbol

$$\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} m_{j,k}(\xi, \eta) r_j(w) r_k(z)$$

which is  $C^\infty$  inside the cone. Fix  $(\xi, \eta)$ . Notice that  $(\xi, \eta)$  is in the support of at most nine of the dyadic parallelograms. Therefore, there exist  $m, n \in \mathbf{Z}$  such that

$$\left| D^\alpha \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} m_{j,k}(\xi, \eta) r_j(w) r_k(z) \right| \leq C \left| D^\alpha \left( \phi \left( 2^{-m} \left( \xi - \frac{\eta}{\tan \theta} \right) \right) \phi \left( 2^{-n} \left( \xi - \frac{\eta}{\tan \theta} \right) \right) \right) \right|.$$

By symmetry, assume that  $m \leq n$ . It is now easy to see that

$$C \left| D^\alpha \left( \phi \left( 2^{-m} \left( \xi - \frac{\eta}{\tan \theta} \right) \right) \phi \left( 2^{-n} \left( \xi - \frac{\eta}{\tan \theta} \right) \right) \right) \right| \leq C(\alpha, \theta) 2^{-m|\alpha|}$$

and

$$\text{dist} \left( (\xi, \eta), \partial P_\theta \right) \leq C 2^m.$$

Applying Theorem B,

$$\begin{aligned} C(r) \left( \int_0^1 \int_0^1 \int_{\mathbf{R}} \left| \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} T_{j,k}(f, g)(x) r_j(w) r_k(z) \right|^r dx dw dz \right)^{\frac{1}{r}} \\ \leq C(r) \left( \int_0^1 \int_0^1 \left\| \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} T_{j,k}(f, g) r_j(w) r_k(z) \right\|_r^r dw dz \right)^{\frac{1}{r}} \\ \leq C(r, \theta) \|f\|_p \|g\|_q \end{aligned}$$

which finishes the proof.  $\square$

## 4. VECTOR-VALUED INEQUALITIES

In this section we focus on vector-valued inequalities for certain bilinear operators in which the Lebesgue spaces  $L^p, L^q$ , and  $L^r$  are replaced by  $L^p(\ell^2), L^q(\ell^2)$ , and  $L^r(\ell^2)$ , respectively. In particular, we are concerned with operators in which the symbols are characteristic functions of the infinite strips and parallelograms discussed in the previous section. Since the infinite strips are just a difference of two half-planes, vector-valued inequalities follow quite simply from the boundedness of the bilinear Hilbert transform [?], [?](see also, [?]). In fact, any sequence of bilinear operators with multipliers consisting of characteristic functions of similar figures satisfies a vector-valued inequality, provided the figures have the same orientation in the plane. Before we further explain this statement and indicate its validity, we state a useful result concerning the bilinear extension of a theorem of Marcinkiewicz and Zygmund.

**Proposition 2.** *Suppose  $0 < p, q, r < \infty$ ,  $1/p + 1/q = 1/r$  and  $T : L^p \times L^q \rightarrow L^r$  is a bounded bilinear operator. Then  $T$  admits an  $\ell^2$ -valued extension, that is there is a constant  $C$  such that for all sequences  $f_k \in L^p(\mathbf{R})$  and  $g_j \in L^q(\mathbf{R})$  we have*

$$(4.18) \quad \left\| \left( \sum_k \sum_j |T(f_k, g_j)|^2 \right)^{1/2} \right\|_r \leq C \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_q.$$

A proof of Proposition ?? can be found in [?] and [?].

**Corollary 5.** *Let  $T$  be as above with symbol  $m(\xi, \eta)$ . If  $T_{j,k}$  has symbol  $m(\xi - c_j, \eta - n_k)$  for real sequences  $\{c_j\}$  and  $\{n_k\}$ , then*

$$(4.19) \quad \left\| \left( \sum_k \sum_j |T_{j,k}(f_k, g_j)|^2 \right)^{1/2} \right\|_r \leq C \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_q.$$

Moreover, if  $\text{supp } (m(\xi - c_j, \eta - n_k)) \subset [2^j, 2^{j+1}] \times [2^k, 2^{k+1}]$  and  $1 < p, q, r < \infty$  such that  $1/p + 1/q = 1/r$ , then

$$(4.20) \quad \left\| \left( \sum_k \sum_j |T_{j,k}(f, g)|^2 \right)^{1/2} \right\|_r \leq C \|f\|_p \|g\|_q.$$

*Proof.* To prove (??), notice that

$$T_{j,k}(f, g)(x) = e^{2\pi i(c_j + n_k)x} T(e^{-2\pi i c_j(\cdot)} f_j(\cdot), e^{-2\pi i n_k(\cdot)} g_k(\cdot))(x).$$

Since  $|e^{-2\pi i c_j(\cdot)} f_j(\cdot)| = |f_j|$  and  $|e^{-2\pi i n_k(\cdot)} g_k(\cdot)| = |g_k|$ ,

$$\left\| \left( \sum_k \sum_j |T_{j,k}(f_k, g_j)|^2 \right)^{1/2} \right\|_r \leq C \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_q$$

which reduces to (??).

To prove (??), let  $\Delta_j f = (\widehat{f} \chi_{[2^j, 2^{j+1}]})^\vee$  and  $\Delta_k g = (\widehat{g} \chi_{[2^k, 2^{k+1}]})^\vee$ . If

$$\text{supp } (m(\xi - c_j, \eta - n_k)) \subset [2^j, 2^{j+1}] \times [2^k, 2^{k+1}],$$

the Littlewood-Paley theorem implies that

$$\begin{aligned} \left\| \left( \sum_k \sum_j |T_{j,k}(f, g)|^2 \right)^{1/2} \right\|_r &= \left\| \left( \sum_k \sum_j |T_{j,k}(\Delta_j(f), \Delta_k(g))|^2 \right)^{1/2} \right\|_r \\ &\leq C(p, q) \left\| \left( \sum_k |\Delta_j(f)|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_j |\Delta_k(g)|^2 \right)^{1/2} \right\|_q \\ &\leq C'(p, q) \|f\|_p \|g\|_q. \end{aligned}$$

□

Let us now consider the nonsmooth decomposition of a one-sided cone into dyadic parallelograms. Let  $\theta_1, \theta_2 \in (0, \pi/2)$  such that  $\theta_1 < \theta_2$  and  $\theta_1 + \theta_2 \neq \pi/4$ . Define the cone

$$C_{\theta_1, \theta_2} = \{(\xi, \eta) : \xi \tan \theta_2 < \eta < \xi \tan \theta_1\}.$$

Let  $\tilde{C}_{\theta_1, \theta_2}(\xi, \eta)$  denote the characteristic function of  $C_{\theta_1, \theta_2}$ . The boundedness of the operator associated with the symbol  $\tilde{C}_{\theta_1, \theta_2}$  follows from Theorem B. For  $i = 1, 2, 3$ , and 4, let  $T_{j,k}^i$  be the bilinear operator associated with the symbol  $m_{j,k}^i$  where

$$m_{j,k}^1(\xi, \eta) = \tilde{C}_{\theta_1, \theta_2}(\xi - 2^k, \eta - 2^j), \quad m_{j,k}^2(\xi, \eta) = \tilde{C}_{\theta_1, \theta_2}(\xi - 2^k, \eta - 2^{j+1})$$

$$m_{j,k}^3(\xi, \eta) = \tilde{C}_{\theta_1, \theta_2}(\xi - 2^{k+1}, \eta - 2^j), \quad m_{j,k}^4(\xi, \eta) = \tilde{C}_{\theta_1, \theta_2}(\xi - 2^{k+1}, \eta - 2^{j+1}).$$

The boundedness of each  $T_{j,k}^i$  follows from Corollary ???. Now, let  $T_{j,k}$  be the operator associated with

$$m_{j,k}(\xi, \eta) = m_{j,k}^1(\xi, \eta) - m_{j,k}^2(\xi, \eta) - m_{j,k}^3(\xi, \eta) + m_{j,k}^4(\xi, \eta).$$

Note that the  $m_{j,k}$ 's have pairwise disjoint supports and

$$\sum_j \sum_k m_{j,k} = C_{\theta_1, \theta_2}$$

It is now easy to establish

$$\left\| \left( \sum_j \sum_k |T_{j,k}(f_j, g_k)|^2 \right)^{1/2} \right\|_r \leq C \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_k |g_k|^2 \right)^{1/2} \right\|_q$$

using

$$|T_{j,k}(f_j, g_k)(x)|^2 \leq \sum_{m=1}^4 \sum_{n=1}^4 |T_{j,k}^m(f_j, g_k)(x)| |T_{j,k}^n(f_j, g_k)(x)|$$

followed by the Cauchy-Schwarz inequality and Corollary ??. Sequences of other polygons can be constructed similarly.

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DEPT. OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA  
E-mail address: mathgr40@math.missouri.edu