

# UNBOUNDEDNESS OF THE BALL BILINEAR MULTIPLIER OPERATOR

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ABSTRACT. For all  $n > 1$ , the characteristic function of the unit ball in  $\mathbb{R}^{2n}$  is not the symbol of a bounded bilinear multiplier operator from  $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$  to  $L^r(\mathbb{R}^n)$  when  $1/p + 1/q = 1/r$  and exactly one of  $p, q$ , or  $r' = r/(r - 1)$  is less than 2.

## 1. INTRODUCTION

We denote the Fourier transform of a function  $f$  on  $\mathbb{R}^n$  by  $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(t)e^{-2\pi i t \cdot \xi} dt$  and its inverse Fourier transform by  $f^\vee(\xi) = \widehat{f}(-\xi)$ . Let  $B$  be the unit ball in  $\mathbb{R}^n$  and  $\chi_A$  the characteristic function of a set  $A$ . The unboundedness of the linear operator

$$T_{\chi_B}(f) = (\widehat{f}\chi_B)^\vee$$

on  $L^p(\mathbb{R}^n)$  when  $p \neq 2$  and  $n > 1$  was established by Fefferman [2].

In this article we provide a variant of Fefferman's result in the bilinear setting. Our arguments also work for multilinear operators. Let  $1 \leq p_1, \dots, p_k \leq \infty$  and  $0 < p < \infty$ . We recall that a bounded function  $m : (\mathbb{R}^n)^k \mapsto \mathbb{C}$  is called a  $k$ -linear multiplier if the  $k$ -linear operator

$$(f_1, \dots, f_k) \mapsto \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} m(\xi_1, \dots, \xi_k) \widehat{f}_1(\xi_1) \dots \widehat{f}_k(\xi_k) e^{2\pi i(\xi_1 + \dots + \xi_k) \cdot x} d\xi_1 \dots d\xi_k$$

initially defined for Schwartz functions  $f_j$  on  $\mathbb{R}^n$  admits a bounded extension

$$(1.1) \quad T_m : L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_k}(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n).$$

In this case we call  $m$  the symbol of  $T_m$ . We will denote by  $\mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^n)$  the set of all  $k$ -linear multipliers  $m$  such that the corresponding operator  $T_m$  satisfies (1.1). The norm of  $m$  in  $\mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^n)$  is defined as the norm of  $T_m$ .

Nontrivial examples of functions in  $\mathcal{M}_{p_1, p_2, p}(\mathbb{R})$  are characteristic functions of half-planes (see [7], [8]) when  $p_1^{-1} + p_2^{-1} = p^{-1} < 3/2$  and characteristic functions of planar ellipses when  $p_1^{-1} + p_2^{-1} = p^{-1}$  and  $2 \leq p_1, p_2, p' < \infty$  (see [4]). Here  $p' = p/(p - 1)$ . It is still an open question whether the results of this paper hold if  $n = 1$ . In this work we show that this is not the case for the characteristic function of the ball in  $\mathbb{R}^{2n}$  if  $1/p + 1/q = 1/r$  and exactly one of  $p, q$ , or  $r'$  is less than 2. We will construct a counterexample when  $n = 2$  and  $r > 2$ . The general result will follow from duality and a multilinear version of de Leeuw's theorem [1].

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2. BILINEARIZATION OF FEFFERMAN'S COUNTEREXAMPLE FOR  $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$ 

For a rectangle  $R$  in  $\mathbb{R}^2$ , let  $R'$  be the union of the two copies of  $R$  adjacent to  $R$  in the direction of its longest side. Hence,  $R \cup R'$  is a rectangle three times as long as  $R$  with the same center. Key to this argument is the following geometric lemma whose proof can be found in [9], page 435 or [3], page 738.

**Lemma 1.** *Let  $\delta > 0$  be given. Then there exists a measurable subset  $E$  of  $\mathbb{R}^2$  and a finite collection of rectangles  $R_j$  in  $\mathbb{R}^2$  such that*

- (1) *The  $R_j$  are pairwise disjoint.*
- (2) *We have  $1/2 \leq |E| \leq 3/2$ .*
- (3) *We have  $|E| \leq \delta \sum_j |R_j|$ .*
- (4) *For all  $j$  we have  $|R'_j \cap E| \geq \frac{1}{12}|R_j|$ .*

Let  $\delta > 0$  and let  $E$  and  $R_j$  be as in Lemma 1. The proof of Lemma 1 implies that there are  $2^k$  rectangles  $R_j$  of dimension  $2^{-k} \times 3 \log(k+2)$ . Here,  $k$  is chosen so that  $k+2 \geq e^{1/\delta}$ . Let  $v_j$  be the unit vector in  $\mathbb{R}^2$  parallel to the longest side of  $R_j$  and in the direction of the set  $E$  relative to  $R_j$ .

**Proposition 1.** *Let  $R$  be a rectangle in  $\mathbb{R}^2$  and let  $v$  be a unit vector in  $\mathbb{R}^2$  parallel to the longest side of  $R$ . Let  $R'$  be as above. Consider the half space  $\mathcal{H}_v$  of  $\mathbb{R}^4$  defined by*

$$\mathcal{H}_v = \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : (\xi + \eta) \cdot v \geq 0\}$$

*Then the following estimate is valid for all  $x \in \mathbb{R}^2$ :*

$$(2.2) \quad \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \chi_{\mathcal{H}_v}(\xi, \eta) \widehat{\chi}_R(\xi) \widehat{\chi}_R(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right| \geq \frac{1}{10} \chi_{R'}(x).$$

*Proof.* We introduce a rotation (i.e. orthogonal matrix)  $\mathcal{O}$  of  $\mathbb{R}^2$  such that  $\mathcal{O}(v) = (1, 0)$ . Setting  $\xi = (\xi_1, \xi_2)$ ,  $\eta = (\eta_1, \eta_2)$  we can write the expression on the left in (2.2) as

$$\begin{aligned} & \left| \iint_{\mathcal{O}^{-1}(\xi + \eta) \cdot v \geq 0} \widehat{\chi}_R(\mathcal{O}^{-1}\xi) \widehat{\chi}_R(\mathcal{O}^{-1}\eta) e^{2\pi i x \cdot \mathcal{O}^{-1}(\xi + \eta)} d\xi d\eta \right| \\ &= \left| \iint_{\xi_1 + \eta_1 \geq 0} \widehat{\chi}_{\mathcal{O}[R]}(\xi) \widehat{\chi}_{\mathcal{O}[R]}(\eta) e^{2\pi i \mathcal{O}x \cdot (\xi + \eta)} d\xi d\eta \right|. \end{aligned}$$

Now the rectangle  $\mathcal{O}[R]$  has sides parallel to the axes, say  $\mathcal{O}[R] = I_1 \times I_2$ . Assume that  $|I_1| > |I_2|$ , i.e. its longest side is horizontal. Let  $H$  be the classical Hilbert transform on the line. Setting  $\mathcal{O}x = (y_1, y_2)$  we can write the last displayed expression as

$$\begin{aligned} & \left| \chi_{I_2}(y_2)^2 \int_{\xi_1 \in \mathbb{R}} \widehat{\chi}_{I_1}(\xi_1) e^{2\pi i y_1 \xi_1} \int_{\eta_1 \geq -\xi_1} \widehat{\chi}_{I_1}(\eta_1) e^{2\pi i y_1 \eta_1} d\eta_1 d\xi_1 \right| \\ &= \chi_{I_2}(y_2) \left| \int_{\xi_1 \in \mathbb{R}} \widehat{\chi}_{I_1}(\xi_1) \frac{1}{2} (I + iH) [\chi_{I_1}(\cdot) e^{2\pi i \xi_1(\cdot)}](y_1) d\xi_1 \right| \\ &= \chi_{I_2}(y_2) \left| \frac{1}{2} (I + iH)(\chi_{I_1})(y_1) \right| = \left| [\chi_{\xi_1 \geq 0} \widehat{\chi}_{I_1 \times I_2}(\xi_1, \xi_2)]^\vee(y_1, y_2) \right|. \end{aligned}$$

Using the result from [3] (Proposition 10.1.2) or [9] (estimate (33), page 453) we deduce that the previous expression is at least

$$\frac{1}{10}\chi_{(I_1 \times I_2)'}(y_1, y_2) = \frac{1}{10}\chi_{(\mathcal{O}[R])'}(\mathcal{O}x) = \frac{1}{10}\chi_{R'}(x).$$

This proves the required conclusion.  $\square$

Next we have the following result concerning bilinear operators on  $\mathbb{R}^2$  of the form

$$T_m(f, g)(x) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} m(\xi_1, \xi_2, \eta_1, \eta_2) \widehat{f}(\xi_1, \xi_2) \widehat{g}(\eta_1, \eta_2) e^{2\pi i x \cdot (\xi_1 + \eta_1, \xi_2 + \eta_2)} d\xi_1 d\xi_2 d\eta_1 d\eta_2.$$

**Lemma 2.** *Let  $v_1, v_2, \dots, v_j, \dots$  be a sequence of unit vectors in  $\mathbb{R}^2$ . Define a sequence of half-spaces  $\mathcal{H}_{v_j}$  in  $\mathbb{R}^4$  as in Proposition 1. Let  $B, B^{*1}, B^{*2}$  be the following sets in  $\mathbb{R}^4$*

$$\begin{aligned} B &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\eta|^2 \leq 1\} \\ B^{*1} &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi + \eta|^2 + |\eta|^2 \leq 1\} \\ B^{*2} &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\xi + \eta|^2 \leq 1\}. \end{aligned}$$

Assume that either  $T_{\chi_B}$  or  $T_{\chi_{B^{*1}}}$  or and  $T_{\chi_{B^{*2}}}$  lies in  $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$  and has norm  $C = C(p, q, r)$ . Then we have the following vector-valued inequality

$$\left\| \left( \sum_j |T_{\chi_{\mathcal{H}_{v_j}}}(f_j, g_j)|^2 \right)^{1/2} \right\|_r \leq C \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_q.$$

for all functions  $f_j$  and  $g_j$ .

*Proof.* We begin with the assumption that  $T_{\chi_B}$  lies in  $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$  for some  $p, q, r > 0$ . Set  $\xi = (\xi_1, \xi_2)$  and  $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ . For  $\rho > 0$  we define sets

$$\begin{aligned} B_\rho &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\eta|^2 \leq 2\rho^2\} \\ B_{j,\rho} &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi - \rho v_j|^2 + |\eta - \rho v_j|^2 \leq 2\rho^2\}. \end{aligned}$$

Note that bilinear multiplier norms are translation and dilation invariant. Easy computations give that

$$\|\chi_{B_{j,\rho}}\|_{\mathcal{M}_{p,q,r}(\mathbb{R}^2)} \leq \|\chi_{B_\rho}\|_{\mathcal{M}_{p,q,r}(\mathbb{R}^2)} = C.$$

The important observation is that  $\chi_{B_{j,\rho}} \rightarrow \chi_{\mathcal{H}_{v_j}}$  pointwise as  $\rho \rightarrow \infty$  and that the multiplier norms of the functions  $\chi_{B_{j,\rho}}$  are bounded above by  $C$ .

Moreover, by the bilinear version of a theorem of Marcinkiewicz and Zygmund ([5], section 9), we have the following inequality for all  $\rho > 0$ .

$$\left\| \left( \sum_j |T_{\chi_{B_\rho}}(f_j, g_j)|^2 \right)^{1/2} \right\|_r \leq C \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_q.$$

Since  $\chi_{B_{j,\rho}} \rightarrow \chi_{\mathcal{H}_{v_j}}$  pointwise as  $\rho \rightarrow \infty$ , we can deduce that

$$\lim_{\rho \rightarrow \infty} T_{\chi_{B_{j,\rho}}}(f, g)(x) = T_{\chi_{\mathcal{H}_{v_j}}}(f, g)(x)$$

for all  $x \in \mathbb{R}^2$  and suitable functions  $f$  and  $g$ . We note that the curvature of the ball  $B$  is used here. By Fatou's lemma we conclude

$$(2.3) \quad \left\| \left( \sum_j |T_{\chi_{\mathcal{H}_{v_j}}}(f_j, g_j)|^2 \right)^{1/2} \right\|_r \leq \liminf_{\rho \rightarrow \infty} \left\| \left( \sum_j |T_{\chi_{B_{j,\rho}}}(f_j, g_j)|^2 \right)^{1/2} \right\|_r.$$

Now, observe the following identity:

$$T_{\chi_{B_{j,\rho}}}(f, g)(x) = e^{4\pi i \rho v_j \cdot x} T_{\chi_{B_\rho}}(e^{-2\pi i \rho v_j \cdot (\cdot)} f, e^{-2\pi i \rho v_j \cdot (\cdot)} g)(x).$$

Using (2.3) and the previous identity gives

$$\begin{aligned} & \left\| \left( \sum_j |T_{\chi_{\mathcal{H}_j}}(f_j, g_j)|^2 \right)^{1/2} \right\|_r \\ & \leq \liminf_{\rho \rightarrow \infty} \left\| \left( \sum_j |e^{4\pi i \rho v_j \cdot (\cdot)} T_{\chi_{B_\rho}}(e^{-2\pi i \rho v_j \cdot (\cdot)} f_j, e^{-2\pi i \rho v_j \cdot (\cdot)} g_j)|^2 \right)^{1/2} \right\|_r \\ & \leq \liminf_{\rho \rightarrow \infty} \left\| \chi_{B_\rho} \right\|_{\mathcal{M}_{p,q,r}} \left\| \left( \sum_j |e^{-2\pi i \rho v_j \cdot (\cdot)} f_j|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_j |e^{-2\pi i \rho v_j \cdot (\cdot)} g_j|^2 \right)^{1/2} \right\|_q \\ & = C \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_p \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_q, \end{aligned}$$

where the last equality follows from the dilation invariance of bilinear multiplier norms.

The proof of the analogous statements for  $T_{B^{*1}}$  and  $T_{B^{*2}}$  is as follows. We introduce sets

$$\begin{aligned} B_\rho^{*1} &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi + \eta|^2 + |\eta|^2 \leq \rho^2\} \\ B_{j,\rho}^{*1} &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi - \rho v_j + \eta|^2 + |\eta|^2 \leq \rho^2\} \\ B_\rho^{*2} &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\xi + \eta|^2 \leq \rho^2\} \\ B_{j,\rho}^{*2} &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\xi + \eta - \rho v_j|^2 \leq \rho^2\}. \end{aligned}$$

Note that both  $B_{j,\rho}^{*1}$  and  $B_{j,\rho}^{*2}$  converge to  $\mathcal{H}_{v_j}$  as  $\rho \rightarrow \infty$ . Using the identities

$$\begin{aligned} T_{\chi_{B_{j,\rho}^{*1}}}(f, g)(x) &= e^{2\pi i \rho v_j \cdot x} T_{\chi_{B_\rho}}(e^{-2\pi i \rho v_j \cdot (\cdot)} f, g)(x) \\ T_{\chi_{B_{j,\rho}^{*2}}}(f, g)(x) &= e^{2\pi i \rho v_j \cdot x} T_{\chi_{B_\rho}}(f, e^{-2\pi i \rho v_j \cdot (\cdot)} g)(x), \end{aligned}$$

we obtain a similar conclusion for the bilinear operators  $T_{\chi_{B^{*1}}}$  and  $T_{\chi_{B^{*2}}}$ .  $\square$

The next ingredient that we will need is a multilinear version of de Leeuw's theorem. For  $1 \leq j \leq k$  we will consider  $\xi_j \in \mathbb{R}^n$ ,  $\eta_j \in \mathbb{R}^m$ . Then the pairs  $(\xi_j, \eta_j) \in \mathbb{R}^{n+m}$ . Also for a function  $f$  on  $\mathbb{R}^n$  and  $g$  on  $\mathbb{R}^m$  we introduce another function  $f \otimes g$  on  $\mathbb{R}^{n+m}$  by setting  $(f \otimes g)(\xi, \eta) = f(\xi)g(\eta)$ .

**Proposition 2.** *Suppose that  $m(\xi_1, \eta_1, \xi_2, \eta_2, \dots, \xi_k, \eta_k) \in \mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^{n+m})$  for some  $1 < p < \infty$ . Then for almost every  $(\xi_1, \dots, \xi_k) \in (\mathbb{R}^n)^k$  the function  $m(\xi_1, \cdot, \xi_2, \cdot, \dots, \xi_k, \cdot)$  lies in  $\mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^m)$ , with norm*

$$\|m(\xi_1, \cdot, \xi_2, \cdot, \dots, \xi_k, \cdot)\|_{\mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^m)} \leq \|m\|_{\mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^{n+m})}.$$

*Proof.* In the proof that follows for simplicity we take  $k = 2$ . The case of a general  $k$  does not present any complications, only notational changes. We also assume that  $m$  is continuous. This assumption may be easily removed by considering convolutions of  $m$  in each variable with smooth approximate identities.

Fix  $f_1, g_1, h_1 \in \mathcal{S}(\mathbb{R}^n)$  and  $f_2, g_2, h_2 \in \mathcal{S}(\mathbb{R}^m)$  with  $\|f_2\|_{p_1} = \|g_2\|_{p_2} = \|h_2\|_{p'} = 1$ . Let

$$M(\xi_1, \xi_2) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} m(\xi_1, \eta_1, \xi_2, \eta_2) \widehat{f}_2(\eta_1) \widehat{g}_2(\eta_2) e^{2\pi i(\eta_1 + \eta_2) \cdot x_2} d\eta_1 d\eta_2 h_2(x_2) dx_2$$

If we can show that  $M \in \mathcal{M}_{p_1, p_2, p}(\mathbb{R}^n)$ , then by Proposition 4 (vi) in [6], we can deduce that  $\|M\|_\infty \leq \|M\|_{\mathcal{M}_{p_1, p_2, p}}$ . Then, by duality, it will follow that  $\|T_m(f_2, g_2)\|_p \leq \|M\|_\infty \leq \|M\|_{\mathcal{M}_{p_1, p_2, p}}$ . We have

$$\begin{aligned}
& |\langle T_M(f_1, g_1), h_1 \rangle| \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} M(\xi_1, \xi_2) \widehat{f}_1(\xi_1) \widehat{g}_1(\xi_2) e^{2\pi i(\xi_1 + \xi_2) \cdot x_1} d\xi_1 d\xi_2 h_1(x_1) dx_1 \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} m(\xi_1, \eta_1, \xi_2, \eta_2) \widehat{f}_2(\eta_1) \widehat{g}_2(\eta_2) e^{2\pi i(\eta_1 + \eta_2) \cdot x_2} d\eta_1 d\eta_2 h_2(x_2) dx_2 \\
&\quad \widehat{f}_1(\xi_1) \widehat{g}_1(\xi_2) e^{2\pi i(\xi_1 + \xi_2) \cdot x_1} d\xi_1 d\xi_2 h_1(x_1) dx_1 \\
&= \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} m(\xi_1, \eta_1, \xi_2, \eta_2) \widehat{f}_1(\xi_1) \widehat{f}_2(\eta_1) \widehat{g}_1(\xi_2) \widehat{g}_2(\eta_2) e^{2\pi i((\xi_1, \eta_1) + (\xi_2, \eta_2)) \cdot (x_1, x_2)} \\
&\quad d(\xi_1, \eta_1) d(\xi_2, \eta_2) h_1(x_1) h_2(x_2) d(x_1, x_2) \\
&= |\langle T_m(f_1 \otimes f_2, g_1 \otimes g_2), h_1 \otimes h_2 \rangle| \\
&\leq \|m\|_{\mathcal{M}_{p_1, p_2, p}(\mathbb{R}^{n+m})} \|f_1 \otimes f_2\|_{p_1} \|g_1 \otimes g_2\|_{p_2} \|h_1 \otimes h_2\|_{p'} \\
&= \|m\|_{\mathcal{M}_{p_1, p_2, p}(\mathbb{R}^{n+m})} \|f_1\|_{p_1} \|f_2\|_{p_1} \|g_1\|_{p_2} \|g_2\|_{p_2} \|h_1\|_{p'} \|h_2\|_{p'} \\
&= \|m\|_{\mathcal{M}_{p_1, p_2, p}(\mathbb{R}^{n+m})} \|f_1\|_{p_1} \|g_1\|_{p_2} \|h_1\|_{p'},
\end{aligned}$$

where the inequality follows from the boundedness of  $T_m$ .  $\square$

The following is the main result of this article.

**Theorem 1.** *Let  $n > 1$  and  $1/p + 1/q = 1/r$  with exactly one of  $p, q$ , or  $r$  less than 2. Let  $B$  be the unit ball in  $\mathbb{R}^{2n}$ . Then  $\chi_B \notin \mathcal{M}_{p, q, r}(\mathbb{R}^n)$ .*

*Proof.* Using Proposition 2 and considering the two dual operators  $T_{\chi_{B^{*1}}}$  and  $T_{\chi_{B^{*2}}}$  of  $T_{\chi_B}$ , it suffices to show that all of these operators are not in  $\mathcal{M}_{p, q, r}(\mathbb{R}^2)$  for  $p, q, r > 2$ . Therefore, we fix  $n = 2$  and  $p, q, r$  satisfying  $p^{-1} + q^{-1} = r^{-1} < 1/2$ . We suppose that  $\chi_B$  is in  $\mathcal{M}_{p, q, r}(\mathbb{R}^2)$  with norm  $C$ .

Suppose that  $\delta > 0$  is given. Let  $E$  and  $R_j$  be as in Lemma 1. Let  $v_j$  be the the unit vector parallel to the longest side of  $R_j$  and pointing in the direction of the set  $E$  relative to  $R_j$ . In the spirit of Fefferman's argument, we estimate  $\sum_j \int_E |T_j(\chi_{R_j}, \chi_{R_j})(x)|^2 dx$  from above and below and arrive to a contradiction. We have

$$\begin{aligned}
& \sum_j \int_E |T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})(x)|^2 dx \\
&\leq |E|^{\frac{r-2}{r}} \left\| \left( \sum_j |T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})|^2 \right)^{1/2} \right\|_r^2 \quad (\text{H\"older's inequality with } r > 2) \\
&\leq C |E|^{\frac{r-2}{r}} \left\| \left( \sum_j |\chi_{R_j}|^2 \right)^{1/2} \right\|_p^2 \left\| \left( \sum_j |\chi_{R_j}|^2 \right)^{1/2} \right\|_q^2 \quad (\text{by Lemma 2}) \\
&= C |E|^{\frac{r-2}{r}} \left( \sum_j |R_j| \right)^{2/r} \quad (\text{by the disjointness of the } R_j\text{s}) \\
&\leq C \delta^{\frac{r-2}{r}} \sum_j |R_j| \quad (\text{Lemma 1}).
\end{aligned}$$

For the reverse inequality we argue as follows:

$$\begin{aligned} \sum_j \int_E |T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})(x)|^2 dx &\geq \sum_j \int_E \left(\frac{1}{10}\chi_{R'_j}(x)\right)^2 dx && \text{(Proposition 1)} \\ &= \frac{1}{100} \sum_j |E \cap R'_j| \\ &\geq \frac{1}{1200} \sum_j |R_j|. && \text{(Lemma 1)} \end{aligned}$$

Putting these two estimates together, we obtain that

$$\frac{1}{1200} \sum_j |R_j| \leq C \delta^{\frac{r-2}{r}} \sum_j |R_j|$$

and therefore

$$\frac{1}{1200} \leq C \delta^{\frac{r-2}{r}}$$

for any  $\delta > 0$ . This is a contradiction since  $r > 2$ . □

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