

Graham's Tree Reconstruction Conjecture and a Waring-Type Problem on Partitions

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Abstract

Suppose G is a tree. Graham's "Tree Reconstruction Conjecture" states that G is uniquely determined by the integer sequence $|G|, |L(G)|, |L(L(G))|, |L(L(L(G)))|, \dots$, where $L(H)$ denotes the line graph of the graph H . Little is known about this question apart from a few simple observations. We show that the number of trees on n vertices which can be distinguished by their associated integer sequences is at least $e^{\Omega((\log n)^{3/2})}$. The proof strategy involves constructing a large collection of caterpillar graphs using partitions arising from the Prouhet-Tarry-Escott problem. We identify, but only partially resolve, an interesting question about representations of integers as sums of k^{th} powers of the parts of integer partitions.

1 Introduction

A conjecture of R. L. Graham (see, e.g., [2]), often referred to as the "Tree Reconstruction Conjecture", states that, if G is a tree, then G is uniquely determined by the sequence of sizes of its iterated line graphs. To make this statement precise, we start with a few definitions. All graphs $G = (V, E)$ are taken to be simple and undirected; a *tree* is an acyclic, connected graph. Given a graph $G = (V, E)$, define the *line graph* $L(G)$ to be a graph with vertex set E , so that, for distinct $e, f \in E$ we have $\{e, f\} \in E(L(G))$ iff $e \cap f \neq \emptyset$, i.e., e and f are incident in G . We denote the j^{th} -iterated line graph by $L^{(j)}(G)$, i.e., $L^{(0)}(G) = G$ and $L^{(j+1)}(G) = L(L^{(j)}(G))$ for $j \geq 1$.

Conjecture 1 (Graham). *For each sequence of natural numbers a_0, a_1, a_2, \dots , all the conditions $|L^{(j)}(G)| = a_j$ for $j \geq 0$ are satisfied by at most one tree G .*

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If G and H are two trees, we say that they are *Graham equivalent* if $|L^{(j)}(G)| = |L^{(j)}(H)|$ for all $j \geq 0$. The corresponding equivalence classes we call *Graham classes*. We can reformulate Conjecture 1 as follows:

Conjecture 2. *For each $n \geq 1$, the number of Graham classes of trees on n vertices equals the number of isomorphism classes of trees on n vertices.*

As shown by Otter ([3]), the number of isomorphism classes of trees on n vertices is $\tilde{\Theta}(\alpha^n)$, where $\alpha = 2.99557658565\dots$, i.e., approximately 3^n . Our main result is the following; though substantially subexponential, the lower bound is at least superpolynomial.

Theorem 1. *The number of Graham classes of trees on n vertices is*

$$\Omega(e^{c(\log n)^{3/2}}).$$

In order to describe the method of proof, we need a few (mostly standard) definitions. For a subset $S \subset V(G)$, $G[S]$ denotes the *induced subgraph* on S , i.e., the graph with vertex set S and edge set $E(G) \cap \binom{S}{2}$; for a vertex $v \in V$, we denote the *neighborhood* $\{w | \{v, w\} \in E(G)\}$ of v by $N_G(v)$, or simply $N(v)$ if G is clear from context. A *path* of length n , denoted P_n , is a tree on the vertex set $\{v_0, \dots, v_n\}$ with an edge between v_j and v_{j+1} for each j , $0 \leq j < n$. A *pendant vertex* in a graph G is a vertex of degree one. A *caterpillar* is a graph obtained from a path by attaching pendant vertices to some of the path vertices. The path from which a caterpillar is built is its *spine*, the vertices on the path of degree greater than two are *joints*, and the leaves attached to the path are *legs*.

The proof proceeds as follows. We construct a collection of caterpillars $\{G_j\}$ on n vertices with distinct sequences $\{|L^{(k)}(G_j)|\}_{k \geq 0}$. To ensure that these sequences differ, we choose the degrees d_1, \dots, d_M of specially selected joints to be a particular class of partitions associated with the Prouhet-Tarry-Escott problem, and leave the rest of the vertices legless. We show that for each k there exists a degree k polynomial $f = f_k$ such that, for some constant $C_{n,k}$ depending on n and k ,

$$|L^k(G_j)| = C_{n,k} + \sum_{i=1}^M f_k(d_i),$$

where $\{d_i\}$ is the degree sequence of the joints of G_j . For ease of notation, if G is a graph and $f : \mathbb{N} \rightarrow \mathbb{N}$ is any function, we write $\hat{f}(G) = \sum_{v \in V(G)} f(\deg(v))$.

In order to complete the proof, we show the following. A sequence \mathbf{a} of nonincreasing positive integers a_1, \dots, a_t is said to *partition* n if $n = \sum_{i=1}^t a_i$; we write $\mathbf{a} \vdash n$.

Theorem 2. *Suppose f is a polynomial of degree d , $d \geq 2$. Then the set $\{f(\lambda) : \lambda \vdash n\}$ has cardinality $\Omega(n^{2(d-2)/3})$.*

We believe the following conjecture to be essentially the strongest version of Theorem 2 possible.

Conjecture 3. *Suppose f is a polynomial of degree d , $d \geq 2$. Then the limit*

$$\lim_{n \rightarrow \infty} \frac{|\{f(\lambda) : \lambda \vdash n\}|}{n^d}$$

exists and is positive.

To actually apply Theorem 2, we will also need to bound from above the ratio of the largest coefficient in the relevant polynomial to its leading coefficient. Much of the work consists of obtaining such bounds; it should be noted, however, that we make little attempt to optimize the resulting expressions other than to simplify exposition.

2 From Caterpillars to Polynomials

The graphs $\{G_i\}_{i \in \mathcal{I}}$ we consider will be caterpillars on n vertices. Given a vertex $v \in V(H)$ for some graph H , define

$$v^k = V(L^k(H)) \setminus V(L^k(H - v)).$$

Note that, if v and w are two vertices of H which are at a distance greater than $2k-2$ from each other, then $v^k \cap w^k = \emptyset$. Therefore, by spacing the joints v_1, \dots, v_t of G_i sufficiently far apart, we may separate their effects on the size of $L^k(G_i)$ from one another. To be precise: given a sequence of positive integers d_1, \dots, d_t and $m > 0$ define $\text{cat}(d_1, \dots, d_t; m)$ to be a caterpillar graph whose spine path of length $(t+1)m$ on the vertex set $v_0, \dots, v_{m(t+1)}$, with d_i legs attached to vertex v_{im} for $1 \leq i \leq t$. Write $S(d; a, b)$ for a star with “central vertex” of degree d to which two disjoint paths are appended at their endvertices: one of length a and one of length b . (See Figure 1.) Then, by considering separately the vertices of $L^k(G_i)$ arising purely from the $P_{m(t+1)}$ spine and those arising from each joint, it is straightforward to see that

$$\begin{aligned} |L^k(\text{cat}(d_1, \dots, d_t; 2k))| &= |L^k(P_{(t+1)(2k)})| + \sum_{j=1}^t (|L^k(S(d_j; k, k))| - |L^k(P_{2k})|) \\ &= (t+1)(2k) - k + 1 - t(k+1) + \sum_{j=1}^t |L^k(S(d_j; k, k))| \\ &= (t+1)(k-1) + 2 + \sum_{j=1}^t |L^k(S(d_j; k, k))| \\ &= k+1 + \sum_{j=1}^t (|L^k(S(d_j; k, k))| + k-1). \end{aligned}$$

Note that

$$2|E(\text{cat}(d_1, \dots, d_t; 2k))| = \sum_{v \in V(\text{cat}(d_1, \dots, d_t; 2k))} \deg(v) = 2k(t+1) + 1 + \sum_{j=1}^t d_j.$$

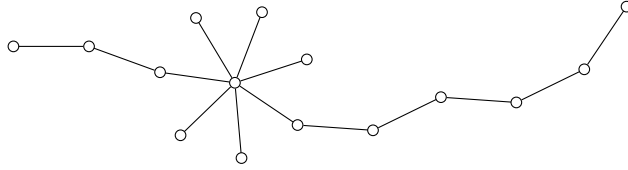


Figure 1: An $S(5; 3, 6)$.

Therefore, if we choose T so that $T \geq t + 1$, then requiring that t is even allows us to append a path of length $Tk - (t+1)(2k) - 3/2$ to one end of the caterpillar, resulting in a graph G_i with

$$\begin{aligned} V(G_i) &= \frac{1}{2} \left[(t+1)(2k) + 1 + \sum_{j=1}^t d_j \right] + 1 + \left(Tk - \frac{(t+1)(2k) + 3}{2} \right) \\ &= \frac{1}{2} \left(\sum_{j=1}^t d_j \right) + Tk \end{aligned}$$

and

$$\begin{aligned} |L^k(\text{cat}(d_1, \dots, d_t; 2k))| &= k + 1 \sum_{j=1}^t (|L^k(S(d_j; k, k))| + k - 1) \\ &\quad + Tk - \frac{(t+1)(2k) + 3}{2} \\ &= Tk - \frac{1}{2} + \sum_{j=1}^t (|L^k(S(d_j; k, k))| - 1). \end{aligned} \quad (1)$$

Then, we need to choose the joint degree sequence of each G_i so that it adds up to the same value, making the G_i have the same size. Therefore, $d_1 + \dots + d_t$ can be thought of as a composition (i.e., ordered partition) of some integer m ; if we can show that the range of $\hat{f}_k(\lambda)$ for $\lambda \vdash m$ has cardinality at least R , then we will have produced R distinct Graham classes. In order to do so, however, the ratio of the largest coefficient of f_k to its leading (highest-degree) coefficient will be important, as we will see in the next section. Therefore, the rest of this section is devoted to bounding this ratio from above.

We consider the graph $S(d; k; k)$. We will show that there exists a single-variable degree k polynomial f_k so that

$$|L^k(S(d; k, k))| - \frac{1}{2} = f_k(d),$$

and then use this fact to construct a large collection of graphs $\{G_i\}_{i \in \mathcal{I}}$ with the same value of t such that, for some fixed k , $\hat{f}_k(G_i) \neq \hat{f}_k(G_j)$ whenever $i \neq j$.

Define $\text{Shadow} : \bigcup_{j=1}^{\infty} \mathcal{P}^{(j)}(V(G)) \rightarrow \mathcal{P}(V(G))$ recursively as follows:

$$\text{Shadow}(S) := \begin{cases} S & \text{if } S \subseteq V(G) \\ \bigcup S & \text{otherwise} \end{cases}$$

Noting that vertices of the k^{th} iterated line graph are unordered pairs of vertices of the $(k-1)^{\text{st}}$ iterated line graph, we see that $\text{Shadow}(v) \subseteq V(G)$ for any $v \in V(L^k(G))$, $k \geq 1$. Enumerate each isomorphism type of connected subgraphs of $S(d; k, k)$ containing the central vertex as $\{H_j\}_{j \in \mathcal{J}}$. Denote the *weight* of a graph H by $\text{wt}(H) := |\{v \in V(L^k(H)) : \text{Shadow}(v) = V(H)\}|$, i.e., the number of vertices in $L^k(H)$ that “involve” all vertices of H . Then we have:

$$|L^k(S(d; k, k))| = |L^k(P_{2k})| + \sum_{j \in \mathcal{J}} \text{wt}(H_j) B_j. \quad (2)$$

where

$$B_j = \begin{cases} \binom{d+2}{a} & \text{if } H_j \cong S(a; 0, 0) \text{ for some } a \geq 1 \\ 2 \binom{d+1}{a} & \text{if } H_j \cong S(a; b, 0) \text{ for some } a \geq 2, b \geq 2 \\ 2 \binom{d}{a} & \text{if } H_j \cong S(a; b, c) \text{ for some } a \geq 1, b \geq 2, c \geq 2, b \neq c \\ \binom{d}{a} & \text{if } H_j \cong S(a; b, b) \text{ for some } a \geq 1, b \geq 2. \end{cases}$$

Note that the H_j all have the form $S(a; b, c)$ for some $0 \leq a, b, c \leq k$, and $\text{wt}(H_j)$ depends only on H_j (and not on n, k , or G). Therefore, (2) combined with (1) provides a count of the vertices of $L^k(G)$ as a linear combination of binomial coefficients whose “numerators” are the degrees of vertices of G and whose “denominators” are at most k , as well as some terms which are linear in k . Since this is a polynomial of degree k , we have our polynomial \hat{f}_k .

We will need an upper bound on the size of the largest coefficient, and a lower bound on the size of the leading coefficient. The coefficients of f_k arise from the $\text{wt}(H_j)$'s and some binomial coefficients, as per (2). We deal with the latter first. As a polynomial in n , $\binom{n}{t} = (n(n-1)\dots(n-t+1))/t!$ has leading coefficient $1/t!$. Each contribution to the coefficient of n^s in the numerator arises by taking the product of some $t-s$ of the numbers $\{1, \dots, t-1\}$; clearly, each such quantity has absolute value at most $(t-1)!$. Since there are at most 2^t elements of $\binom{[t-1]}{[t-s]}$,

$$[n^r] \binom{n}{t} \leq \frac{2^t (t-1)!}{t!} = \frac{2^t}{t}. \quad (3)$$

for each r , $1 \leq r \leq t$.

Lemma 3. *If G is a d -regular graph, then $L^{(k)}(G)$ is $(2^k d - 2^{k+1} + 2)$ -regular.*

Proof. We proceed by induction. The base case is almost immediate: Given an edge $e \in E(G)$, its endvertices each have degree k . Therefore e is incident to $d-1 + d-1 = 2d-2$ edges f in G , whence the degree of each vertex in $L(G)$ is $2d-2$. Since $2d-2 = 2^1 d - 2^2 + 2$, we are done. Now, suppose that $L^{(k)}(G)$ is

$(2^k d - (2^{k+1} - 2))$ -regular. By the base case, $L^{(k+1)}(G)$ is $(2 \cdot 2^k d - 2 \cdot 2^{k+1} + 4 - 2)$ -regular. However,

$$2 \cdot 2^k d - 2 \cdot 2^{k+1} + 4 - 2 = 2^{k+1} d - 2^{k+2} + 2.$$

□

Corollary 4. For $k \geq 1$ and $d \geq 3$, $|L^{(k)}(S(d; a, b))| < (d + a + b)^k 2^{k^2/2}$.

Proof. Let S_d denote the d -star, i.e., $S_d = S(d; 0, 0)$. It is easy to see that $L(S_d) = K_d$, which is $(d - 1)$ -regular. Also, for any r -regular graph H ,

$$|L(H)| = |E(H)| = \frac{r|H|}{2},$$

Then, by Lemma 3,

$$\begin{aligned} |L^{(k)}(S_d)| &= |L^{(k-1)}(K_d)| \\ &= |K_d| \prod_{j=1}^{k-1} \frac{2^j(d-3) + 2}{2} \\ &= d \prod_{j=1}^{k-1} (2^{j-1}(d-3) + 1). \end{aligned}$$

Since $j \geq 1$, we have $2^{j-1}(d-3) + 1 \leq 2^{j-1}d$, so

$$\begin{aligned} |L^{(k)}(S_d)| &= d \prod_{j=1}^{k-1} (2^{j-1}(d-3) + 1) \\ &\leq d \prod_{j=1}^{k-1} 2^{j-1}d \\ &< d^k 2^{k^2/2}. \end{aligned}$$

Now, since $|L(S(d; a, b))| = d + a + b$, we have $L(S(d; a, b)) \subseteq K_{d+a+b}$. Therefore, we may simply repeat the above calculation for S_{d+a+b} , and the result follows. □

We now need an upper bound on the number of terms present in expression (2). Recall that the H_j range over isomorphism classes of graphs which occur in the shadow of nodes in the k^{th} iterated line graph. Below we show that these isomorphism classes have at most $k + 1$ vertices.

Lemma 5. For any graph H and $k \geq 0$, $|\text{Shadow}(v)| \leq k + 1$ for all $v \in V(L^{(k)}(H))$.

Proof. We represent each vertices of $H, L(H), \dots, L^{(k)}(H)$ by a complete $(k+1)$ -level binary tree as follows. The root node is v . Since $v \in L^{(k)}(H)$, v is an unordered pair $\{u, w\}$ of vertices from $L^{(k-1)}(H)$. Then u and w are the children of v . Similarly, u 's children are its two constituent nodes (since $u \in (V(L^{(k-2)}(H)))$) and $w = \{a, b\}$ has children a and b . We proceed in the obvious way until a vertex of G is reached, whereupon we do not give that node any children.

We now argue the following:

Claim 1. *If $G = (V, E)$ is a graph, then for $v \in V(L^t(G))$, $t \geq 1$, if A is a child of v in a binary tree constructed as above, then $|\text{Shadow}(v) \setminus \text{Shadow}(A)| \leq 1$.*

We proceed by induction. First, the claim is obvious when $t = 1$. For $t = 2$: Let $v \in V(L^2(G))$. Then $v = \{A, B\}$ where $A = \{e_1, e_2\}$ and $B = \{e_2, e_3\}$ with $e_i \in E(G)$. Furthermore, $e_1 = \{a, b\}$, $e_2 = \{b, c\}$, and $e_3 = \{c, d\}$ for some $a, b, c, d \in V(G)$, since e_1 and e_2 as well as e_2 and e_3 must be incident, i.e., share a vertex. Therefore, $\text{Shadow}(v) = \{a, b, c, d\}$ and $\text{Shadow}(A) = \{a, b, c\}$, so $|\text{Shadow}(v) \setminus \text{Shadow}(A)| = |\{d\}| = 1$.

Now, suppose the claim is true for all $1 \leq t \leq s$, $s > 2$, and let $v \in L^{(s+1)}(G)$, with $v = \{A, B\}$, $A, B \in L^{(s)}(G)$. Note that $A \cap B$ is a single vertex w of $L^{(s-1)}(G)$, since A and B are unordered pairs of vertices from $L^{(s-1)}(G)$ which must intersect. Let $S = \text{Shadow}(w)$. Then, by the inductive hypothesis and the fact that w is a child of A (so, in particular, $S \subseteq \text{Shadow}(A)$), either $\text{Shadow}(A) = S$ or $\text{Shadow}(A) = S \cup \{a\}$ for some $a \in V(G) \setminus S$. Similarly, $\text{Shadow}(B) = S$ or $\text{Shadow}(B) = S \cup \{b\}$ for some $b \in V(G) \setminus S$. Then we have four cases:

(i) $\text{Shadow}(A) = S$ and $\text{Shadow}(B) = S$: Then

$$\begin{aligned} \text{Shadow}(v) \setminus \text{Shadow}(A) &= \text{Shadow}(A \cup B) \setminus \text{Shadow}(A) \\ &= [\text{Shadow}(A) \cup \text{Shadow}(B)] \setminus \text{Shadow}(A) \\ &= (S \cup S) \setminus S = \emptyset. \end{aligned}$$

(ii) $\text{Shadow}(A) = S \cup \{a\}$ and $\text{Shadow}(B) = S$: Then

$$\begin{aligned} \text{Shadow}(v) \setminus \text{Shadow}(A) &= \text{Shadow}(A \cup B) \setminus \text{Shadow}(A) \\ &= [\text{Shadow}(A) \cup \text{Shadow}(B)] \setminus \text{Shadow}(A) \\ &= (S \cup \{a\} \cup S) \setminus (S \cup \{a\}) = \emptyset. \end{aligned}$$

(iii) $\text{Shadow}(A) = S$ and $\text{Shadow}(B) = S \cup \{b\}$: Then

$$\begin{aligned} \text{Shadow}(v) \setminus \text{Shadow}(A) &= \text{Shadow}(A \cup B) \setminus \text{Shadow}(A) \\ &= [\text{Shadow}(A) \cup \text{Shadow}(B)] \setminus \text{Shadow}(A) \\ &= (S \cup S \cup \{b\}) \setminus S = \{b\}. \end{aligned}$$

(iv) $\text{Shadow}(A) = S \cup \{a\}$ and $\text{Shadow}(B) = S \cup \{b\}$: Then

$$\begin{aligned} \text{Shadow}(v) \setminus \text{Shadow}(A) &= \text{Shadow}(A \cup B) \setminus \text{Shadow}(A) \\ &= [\text{Shadow}(A) \cup \text{Shadow}(B)] \setminus \text{Shadow}(A) \\ &= (S \cup \{a\} \cup S \cup \{b\}) \setminus (S \cup \{a\}) \\ &= (S \cup \{a, b\}) \setminus (S \cup \{a\}), \end{aligned}$$

which is $\{b\}$ if $a \neq b$ and \emptyset otherwise.

In each case, the Claim is verified, and the result follows by induction on t . Then the Lemma follows by induction on k and the observation that $|\text{Shadow}(v)| = 1$ for $v \in V(G)$. \square

Theorem 6. *An upper bound on the maximum coefficient (in absolute value) of f_k is $(k+3)^{k+2} 2^{k^2/2+k+4}$.*

Proof. Let the maximum coefficient of f_k be C . Going back to expression (2), we see that

$$C \leq |\mathcal{J}| \cdot \max_{j \in \mathcal{J}} \text{wt}(H_j) \cdot \max_{j \in \mathcal{J}} B_j.$$

To bound the first factor, we count the isomorphism classes of graphs on $\leq k+1$ vertices (by Lemma 5) which can be embedded into $S(d; k, k)$ and contain the star vertex. Suppose $H_j = S(a; b, c)$; then $|H_j| = a + b + c + 1$. Therefore, an upper bound for the number of elements of \mathcal{J} is the number of nonnegative integer solutions to $a + b + c + 1 \leq k + 1$, i.e., the number of nonnegative integer solutions to $a + b + c + d = k$. This is easily seen to be $\binom{k+3}{3}$.

To bound the second factor, we employ Corollary 4. In particular, writing $H_j = S(d_j; a_j, b_j)$,

$$\begin{aligned} \max_{j \in \mathcal{J}} \text{wt}(H_j) &= \max_{j \in \mathcal{J}} \text{wt}(S(d_j; a_j, b_j)) \\ &< \max_{j \in \mathcal{J}} (d_j + a_j + b_j)^k 2^{k^2/2} \\ &\leq (k+1)^k 2^{k^2/2} \end{aligned}$$

by Lemma 5.

To bound the third factor, we refer to (3) and the definition of B_j . The coefficients of B_j are bounded by

$$2 \cdot \frac{2^{k+3}}{k+3} = \frac{2^{k+4}}{k+3},$$

since $d \leq k+1$.

Putting the pieces together, we see that

$$C \leq \binom{k+3}{3} \cdot (k+1)^k 2^{k^2/2} \cdot \frac{2^{k+4}}{k+3}$$

$$\begin{aligned} &\leq \frac{1}{6}(k+3)^2(k+1)^k 2^{k^2/2+k+4} \\ &< (k+3)^{k+2} 2^{k^2/2+k+4}. \end{aligned}$$

□

Corollary 7. *An upper bound on the ratio of the maximum coefficient to the leading coefficient of f_k is*

$$(k+1)!(k+3)^{k+2} 2^{k^2/2+k+4} < (k+3)^{2k+3} 2^{k^2/2+k+4}.$$

Proof. By the observations preceding (3), we can take $\frac{1}{(k+1)!}$ as a lower bound on the leading coefficient. □

3 Sums of Powers of Parts

For a (multi)set A of integers, let $S_k(A) = \sum_{a \in A} a^k$, and let $A + t = \{a + t : a \in A\}$, for $t \in \mathbb{Z}$. Now, define the sequence of sets \mathcal{T}_j as follows:

1. $\mathcal{T}_0 = \emptyset$
2. $\mathcal{T}_{j+1} = \mathcal{T}_j \cup (\{0, \dots, 2^j - 1\} \setminus \mathcal{T}_j + 2^j)$

Furthermore, define $\overline{\mathcal{T}}_j = \{0, \dots, 2^{j+1} - 1\} \setminus \mathcal{T}_j$. In other words, the set \mathcal{T}_k consists of those integers in $\{0, \dots, 2^k - 1\}$ the sum of whose binary digits is odd, and $\overline{\mathcal{T}}_k$ consists of the set of integers in $\{0, \dots, 2^k - 1\}$ the sum of whose binary digits is even.

It has been known since 1851 ([4]) that

$$S_k(\mathcal{T}_r) = S_k(\overline{\mathcal{T}}_r) \tag{4}$$

when $k < r$, i.e., the pair $(\mathcal{T}_r, \overline{\mathcal{T}}_r)$ provides a solution to the degree- k Prouhet-Tarry-Escott problem (q.v. [1]). When $k = r$, however, these two quantities are no longer equal.

Proposition 8. $S_k(\mathcal{T}_k) - S_k(\overline{\mathcal{T}}_k) = (-1)^{k+1} k! 2^{\binom{k}{2}}$ for $k \geq 1$. Furthermore,

$$|S_{k+j}(\mathcal{T}_k) - S_{k+j}(\overline{\mathcal{T}}_k)| \leq (k+j)! 2^{\binom{k+j+1}{2}}$$

for $j \geq 0$.

Proof. We begin with the first statement, and proceed by induction. For $k = 1$,

$$S_1(\mathcal{T}_1) - S_1(\overline{\mathcal{T}}_1) = 1^1 - 0^1 = 1 = (-1)^{1+1} 1! 2^{\binom{1}{2}}.$$

Suppose the statement is true for $k - 1$. Then we may write

$$S_k(\mathcal{T}_k) - S_k(\overline{\mathcal{T}}_k) = S_k(\mathcal{T}_{k-1}) - S_k(\overline{\mathcal{T}}_{k-1}) + S_k(2^k + \overline{\mathcal{T}}_{k-1}) - S_k(2^k + \mathcal{T}_{k-1})$$

$$\begin{aligned}
&= S_k(\mathcal{T}_{k-1}) - S_k(\overline{\mathcal{T}}_{k-1}) + \sum_{j=0}^k \binom{k}{j} 2^{kj} S_{k-j}(\overline{\mathcal{T}}_{k-1}) \\
&\quad - \sum_{j=0}^k \binom{k}{j} 2^{kj} S_{k-j}(\mathcal{T}_{k-1})
\end{aligned}$$

by the binomial theorem. Therefore,

$$\begin{aligned}
S_k(\mathcal{T}_k) - S_k(\overline{\mathcal{T}}_k) &= \sum_{j=1}^k \binom{k}{j} 2^{kj} S_{k-j}(\overline{\mathcal{T}}_{k-1}) - \sum_{j=1}^k \binom{k}{j} 2^{kj} S_{k-j}(\mathcal{T}_{k-1}) \\
&= k2^k (S_{k-1}(\overline{\mathcal{T}}_{k-1}) - S_{k-1}(\mathcal{T}_{k-1})),
\end{aligned}$$

since, by (4), all terms with $j > 1$ are zero. Applying the inductive hypothesis then,

$$\begin{aligned}
S_k(\mathcal{T}_k) - S_k(\overline{\mathcal{T}}_k) &= -k2^k \cdot (-1)^k (k-1)! 2^{\binom{k-1}{2}} \\
&= (-1)^{k+1} k! 2^{\binom{k}{2}}.
\end{aligned}$$

Now, we show by a double induction that the claimed bound holds for each $j \geq 0$. Denote by $A_j(k)$ the quantity $|S_{k+j}(\mathcal{T}_k) - S_{k+j}(\overline{\mathcal{T}}_k)|$. The base case of the outer induction is clear, since

$$A_0(k) = k! 2^{\binom{k}{2}} \leq k! 2^{\binom{k+1}{2}}.$$

Now, suppose the statement is true up to j . The base case of the inner induction is trivial, since

$$A_j(1) = |S_{j+1}(\mathcal{T}_1) - S_{j+1}(\overline{\mathcal{T}}_1)| = 1^{j+1} - 0^{j+1} = 1,$$

which is clearly bounded above by $(j+1)! 2^{\binom{j+2}{2}}$. So, suppose that the claimed bound holds for $j+1$ and any argument less than k . Then we have

$$\begin{aligned}
A_j(k) &= |S_{k+j}(\mathcal{T}_{k-1}) - S_{k+j}(\overline{\mathcal{T}}_{k-1}) + S_{k+j}(2^k + \overline{\mathcal{T}}_{k-1}) - S_{k+j}(2^k + \mathcal{T}_{k-1})| \\
&= \left| S_{k+j}(\mathcal{T}_{k-1}) - S_{k+j}(\overline{\mathcal{T}}_{k-1}) + \sum_{r=0}^{k+j} \binom{k+j}{r} 2^{kr} S_{k+j-r}(\overline{\mathcal{T}}_{k-1}) \right. \\
&\quad \left. - \sum_{r=0}^{k+j} \binom{k+j}{r} 2^{kr} S_{k+j-r}(\mathcal{T}_{k-1}) \right| \\
&= \left| \sum_{r=1}^{k+j} \binom{k+j}{r} 2^{kr} S_{k+j-r}(\overline{\mathcal{T}}_{k-1}) - \sum_{r=1}^{k+j} \binom{k+j}{r} 2^{kr} S_{k+j-r}(\mathcal{T}_{k-1}) \right| \\
&\leq \sum_{r=1}^{k+j} \binom{k+j}{r} 2^{kr} A_{j-r+1}(k-1).
\end{aligned}$$

Therefore, applying both inductive hypotheses,

$$\begin{aligned}
A_j(k) &\leq \sum_{r=1}^{k+j} \binom{k+j}{r} 2^{kr} A_{j-r+1}(k-1) \\
&\leq \sum_{r=1}^{k+j} \frac{\prod_{i=0}^{r-1} (k+j-i)}{r!} 2^{kr} (k+j-r)! 2^{\binom{k+j-r}{2}} \\
&= \sum_{r=1}^{k+j} \frac{1}{r!} (k+j)! 2^{(k+j-r+1)+kr} \\
&= (k+j)! \sum_{r=1}^{k+j} \frac{1}{r!} 2^{(k+j+1)-rj-\binom{r}{2}} \\
&\leq (k+j)! 2^{\binom{k+j+1}{2}} \sum_{r=1}^{k+j} \frac{1}{r!} 2^{-rj} \\
&< (k+j)! 2^{\binom{k+j+1}{2}} \sum_{r=1}^{\infty} \frac{2^{-rj}}{r!} \\
&= (k+j)! 2^{\binom{k+j+1}{2}} (e^{2^{-j}} - 1) \\
&< (k+j)! 2^{\binom{k+j+1}{2}}
\end{aligned}$$

for $j \geq 1$. □

Define the sequence $\mathbf{W}(k; r, s, t)$ as follows. Let \mathbf{T}_j be the sequence consisting of the elements of \mathcal{T}_j in increasing order; let $\overline{\mathbf{T}}_j$ be the sequence consisting of the elements of $\overline{\mathcal{T}}_j$ in increasing order. Then, for $r \geq 0$ and $s \geq t \geq 0$,

$$\begin{aligned}
\mathbf{W}(k; r, s, t) &= (\overline{\mathbf{T}}_k)^r (\mathbf{T}_k)^s \left(\prod_{j=1}^t (\overline{\mathbf{T}}_k + j2^k) \right) (\mathbf{T}_k + (t+1)2^k) \\
&\quad \cdot \prod_{j=1}^{s-t} (\overline{\mathbf{T}}_k + (j+t+1)2^k) \prod_{j=1}^r (\mathbf{T}_k + (j+s+1)2^k)
\end{aligned}$$

where addition is interpreted componentwise, products of sequences are interpreted as concatenation, and the empty product is interpreted as the empty sequence. So, for example,

$$\mathbf{W}(2; 2, 2, 1) = 0, 3, 0, 3, 1, 2, 1, 2, 4, 7, 9, 10, 12, 15, 17, 18, 21, 22$$

and

$$\begin{aligned}
\mathbf{W}(3; 3, 1, 1) &= 0, 3, 5, 6, 0, 3, 5, 6, 0, 3, 5, 6, 1, 2, 4, 7, 8, 11, 13, 14, \\
&\quad 17, 10, 20, 23, 25, 26, 28, 31, 33, 34, 36, 39, 41, 42, 44, 47
\end{aligned}$$

It is easy to verify that the sum of the elements of \mathcal{T}_k (or $\overline{\mathcal{T}}_k$) is $B = 4^{k-1} - 2^{k-2}$ for $k \geq 2$. Therefore, for $k \geq 2$, $\mathbf{W}(k; r, s, t)$ is a partition of

$$\begin{aligned}
& Br + Bs + \left(\sum_{j=1}^t B + 2^{k-1} j 2^k \right) + (B + (t+1)2^k) + \left(\sum_{j=1}^{s-t} B + 2^{k-1} (t+j+1) 2^k \right) \\
& \quad + \left(\sum_{j=1}^r B + 2^{k-1} (s+j+1) 2^k \right) \\
& = B(r + s + t + 1 + s - t + r) \\
& \quad + 2^{2k-1} \left(t + 1 + \sum_{j=1}^t j + \sum_{j=1}^{s-t} (t+j+1) + \sum_{j=1}^r (s+j+1) \right) \\
& = B(2r + 2s + 1) \\
& \quad + 2^{2k-1} \left(t + 1 + (t+1)(s-t) + (s+1)r + \sum_{j=1}^t j + \sum_{j=1}^{s-t} j + \sum_{j=1}^r j \right) \\
& = B(2r + 2s + 1) + 2^{2k-1} ((t+1)(s-t+1) + (s+1)r \\
& \quad + \frac{t(t+1)}{2} + \frac{(s-t)(s-t+1)}{2} + \frac{r(r+1)}{2}) \\
& = (4^{k-1} - 2^{k-2})(2r + 2s + 1) + 2^{2k-1}(rs + s^2 + r^2 + 3s + 3r) \\
& = 4^{k-1}(8r + 8s + 1 + 2rs + 2s^2 + 2r^2) - 2^{k-2}(2r + 2s + 1).
\end{aligned}$$

Also, the number of parts in the partition represented by $\mathbf{W}(k; r, s, t)$ is $2^{k-1}(2r + 2s + 1)$.

Define \mathcal{W}_j^k for $0 \leq j < s(s+1)/2$ to be the j -th element of the sequence:

$$\begin{aligned}
& \mathbf{W}(k; 0, s, 0) \rightarrow \mathbf{W}(k; 0, s, 1) \rightarrow \mathbf{W}(k; 0, s, 2) \rightarrow \cdots \mathbf{W}(k; 0, s, s) \\
& \rightarrow \mathbf{W}(k; 1, s-1, 0) \rightarrow \mathbf{W}(k; 1, s-1, 1) \rightarrow \cdots \mathbf{W}(k; 1, s-1, s-1) \\
& \rightarrow \mathbf{W}(k; 2, s-2, 0) \rightarrow \mathbf{W}(k; 2, s-2, 1) \rightarrow \cdots \mathbf{W}(k; 2, s-2, s-2) \\
& \quad \vdots \\
& \rightarrow \mathbf{W}(k; s-1, 1, 0) \rightarrow \mathbf{W}(k; s-1, 1, 1) \\
& \rightarrow \mathbf{W}(k; s, 0, 0).
\end{aligned}$$

Note that each of these \mathcal{W}_j^k is a composition (i.e., ordered partition) of $4^{k-1}(2s^2 + 8s + 1) - 2^{k-2}(2s + 1)$ of length $2^{k-1}(2s + 1)$.

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a (multi)set or sequence $S \subset \mathbb{R}$, we write

$$f(S) = \sum_{s \in S} f(s).$$

Lemma 9. For a polynomial f of degree $d \geq k \geq 1$, there is a polynomial g of degree $d - k$ so that

$$f(\mathcal{T}_k + t) - f(\overline{\mathcal{T}}_k + t) = g(t).$$

Furthermore, if C is the lead coefficient of f , then the lead coefficient of g has absolute value $C \binom{d}{k} k! 2^{\binom{k}{2}}$, and the sum of the rest of the coefficients is at most

$$C' e d^2 d! 2^{\binom{d+1}{2}},$$

where C' is the largest non-leading coefficient of f .

Proof. Suppose $f(x) = \sum_{j=0}^d a_j x^j$. We may write

$$\begin{aligned} f(\mathcal{T}_k + t) &= \sum_{x \in \mathcal{T}_k} f(x + t) \\ &= \sum_{x \in \mathcal{T}_k} \sum_{j=0}^d a_j (x + t)^j \\ &= \sum_{x \in \mathcal{T}_k} \sum_{j=0}^d \sum_{i=0}^j a_j \binom{j}{i} x^i t^{j-i}. \end{aligned}$$

Similarly,

$$f(\overline{\mathcal{T}}_k + t) = \sum_{x \in \overline{\mathcal{T}}_k} \sum_{j=0}^d \sum_{i=0}^j a_j \binom{j}{i} x^i t^{j-i}.$$

Therefore,

$$\begin{aligned} f(\mathcal{T}_k + t) - f(\overline{\mathcal{T}}_k + t) &= \sum_{j=0}^d \sum_{i=0}^j a_j \binom{j}{i} t^{j-i} \left(\sum_{x \in \mathcal{T}_k} x^i - \sum_{x \in \overline{\mathcal{T}}_k} x^i \right) \\ &= \sum_{j=0}^d \sum_{i=k}^j a_j \binom{j}{i} t^{j-i} \left(\sum_{x \in \mathcal{T}_k} x^i - \sum_{x \in \overline{\mathcal{T}}_k} x^i \right) \\ &= \sum_{q=0}^{d-k} t^q \sum_{j=q+k}^d a_j \binom{j}{q} \left(\sum_{x \in \mathcal{T}_k} x^{j-q} - \sum_{x \in \overline{\mathcal{T}}_k} x^{j-q} \right), \end{aligned}$$

where the second equality follows from the fact that the pair $\{\mathcal{T}_k, \overline{\mathcal{T}}_k\}$ is a solution to the Prouhet-Tarry-Escott problem of any order $i < k$. To complete the proof, we need only show that the coefficient c_{d-k} of t^{d-k} is nonzero. However,

$$c_{d-k} = \binom{d}{k} \left(\sum_{x \in \mathcal{T}_k} x^k - \sum_{x \in \overline{\mathcal{T}}_k} x^k \right)$$

$$= (-1)^{k+1} \binom{d}{k} k! 2^{\binom{k}{2}} \neq 0,$$

by Proposition 8. For the second part of the Lemma, we note that the sum of the non-leading coefficients of g is at most the largest non-leading coefficient of f times

$$\begin{aligned} \left| \sum_{q=0}^{d-k-1} \sum_{j=q+k}^d \binom{j}{q} (S_{j-q}(\mathcal{T}_k) - S_{j-q}(\overline{\mathcal{T}}_k)) \right| &\leq \sum_{q=0}^{d-k-1} \sum_{j=q+k}^d \binom{j}{q} (j-q)! 2^{\binom{j-q+1}{2}} \\ &= \sum_{q=0}^{d-k-1} \sum_{j=q+k}^d \frac{j!}{q!} 2^{\binom{j-q+1}{2}} \\ &< d \cdot d! \sum_{q=0}^{d-k-1} \frac{1}{q!} 2^{\binom{d-q+1}{2}} \\ &< ed^2 d! 2^{\binom{d+1}{2}}, \end{aligned}$$

where the first inequality appeals to the second part of Proposition 8. \square

Corollary 10. *For $r \geq 0$ and $s > t \geq 0$ a polynomial f of degree $d \geq k+1 \geq 3$, there is a polynomial g of degree $d-k-1$ so that*

$$f(\mathbf{W}(k; r, s, t+1)) - f(\mathbf{W}(k; r, s, t)) = g((t+1)2^k)$$

Furthermore, if C is the lead coefficient of f , then the lead coefficient of g has absolute value $C \binom{d}{k+1} (k+1)! 2^{\binom{k+1}{2}}$, and the sum of the rest of the coefficients is at most

$$C' ed^2 d! 2^{\binom{d+1}{2}},$$

where C' is the largest non-leading coefficient of f . In addition, for $r \leq s-1$,

$$f(\mathbf{W}(k; r+1, s-1, 0)) - f(\mathbf{W}(k; r, s, s)) = g(0)$$

Proof. It is easy to see that $\mathbf{W}(k; r, s, t+1)$ can be obtained from $\mathbf{W}(k; r, s, t)$ by replacing the subsequence $\mathbf{T}_{k+1} + (t+1)2^k$ by the sequence $\overline{\mathbf{T}}_{k+1} + (t+1)2^k$. The first conclusion then follows immediately from Lemma 9. The second statement follows from Lemma 9 and the observation that $\mathbf{W}(k; r+1, s-r-1, 0)$ can be obtained from $\mathbf{W}(k; r, s-r, s-r)$ by replacing the subsequence \mathbf{T}_{k+1} with $\overline{\mathbf{T}}_{k+1}$. \square

Theorem 11. *Given a polynomial f of degree d , $d \geq 2$, let C denote the lead coefficient of f , C' its largest coefficient, and $\alpha = \lceil C'/C \rceil$. Then the set $\{f(\lambda) : \lambda \vdash n\}$ has cardinality $\Omega(n^{2(d-2)/3})$. More precisely, for $n \geq 42d^4 \alpha^3 / 2^3 2^{3d^2/5+11d/10}$,*

$$|\{f(\lambda) : \lambda \vdash n\}| > 2^{-2d^3/5-1800} n^{2(d-2)/3}.$$

Proof. Let $R_k = \lceil C'4ed^22^{\binom{d+1}{2}-\binom{k+2}{2}}(d-k-1)!/C \rceil$. Note that, for $1 \leq k \leq d-1$,

$$\frac{R_{k-1}}{R_k} \geq \frac{1}{2}2^{\binom{k+2}{2}-\binom{k+1}{2}} \frac{(d-k)!}{(d-k-1)!} = 2^k(d-k).$$

The sequence of changes in value of $f(\mathcal{W}_j^k + \alpha R_k 2^k)$, $0 \leq j < s(s+1)/2$, $\alpha \geq 1$, is given by

$$\begin{aligned} &g((\alpha R_k + 1)2^k) \rightarrow g((\alpha R_k + 2) \cdot 2^k) \rightarrow g((\alpha R_k + 3) \cdot 2^k) \rightarrow \cdots g((\alpha R_k + s)2^k) \\ &\rightarrow g(\alpha R_k 2^k) \rightarrow g((\alpha R_k + 1)2^k) \rightarrow g((\alpha R_k + 2) \cdot 2^k) \rightarrow \cdots g((\alpha R_k + s - 1)2^k) \\ &\rightarrow g(\alpha R_k 2^k) \rightarrow g((\alpha R_k + 1)2^k) \rightarrow g((\alpha R_k + 2) \cdot 2^k) \rightarrow \cdots g((\alpha R_k + s - 2)2^k) \\ &\vdots \\ &\rightarrow g(\alpha R_k 2^k) \rightarrow g((\alpha R_k + 1)2^k) \\ &\rightarrow g(\alpha R_k 2^k). \end{aligned}$$

Each $\mathcal{W}_j^k + \alpha R_k 2^k$ is a partition of

$$4^{k-1}(2s^2 + 8s + 1) + 2^{k-2}(\alpha R_k 2^{k+1} - 1)(2s + 1)$$

of length $2^{k-1}(2s + 1)$. Note that

$$\begin{aligned} |g(m2^k)| &= C \binom{d}{k+1} (k+1)! 2^{\binom{k+1}{2}} (m2^k)^{d-k-1} \pm C'ed^2d!2^{\binom{d+1}{2}} (m2^k)^{d-k-2} \\ &= (m2^k)^{d-k-2} \left(C \binom{d}{k+1} (k+1)! 2^{\binom{k+1}{2}} m2^k \pm C'ed^2d!2^{\binom{d+1}{2}} \right). \end{aligned}$$

Then, if $m \geq R_k$, then

$$m \geq R_k > \frac{C'}{C} 4ed^22^{\binom{d+1}{2}-\binom{k+2}{2}}(d-k-1)!,$$

so

$$|g(m2^k)| = m^{d-k-1} C \binom{d}{k+1} (k+1)! 2^{dk-\binom{k+1}{2}} \left(1 \pm \frac{1}{2} \right). \quad (5)$$

Let λ_j^k , $1 \leq j \leq s(s+1)/2$, denote the integer partition induced by $\mathcal{W}_{j+1}^k + \alpha R_k 2^k$ as above. Then $\{\lambda_j^k\}_{j=1}^{s(s+1)/2}$ is a sequence of partitions of $4^{k-1}(2s^2 + 8s + 1) + 2^{k-2}(R_k 2^{k+1} - 1)(2s + 1)$ so that

$$f(\lambda_{j+1}^k) - f(\lambda_j^k) \geq (\alpha R_k)^{d-k-1} C \binom{d}{k+1} (k+1)! 2^{dk-\binom{k+1}{2}-1}$$

for $1 \leq j \leq s(s+1)/2 - 1$. Furthermore,

$$f(\lambda_{s(s+1)/2}^k) - f(\lambda_1^k) \leq (s + \alpha R_k)^{d-k-1} s(s+1) C \binom{d}{k+1} (k+1)! 2^{dk-\binom{k+1}{2}-1}$$

$$\leq (s + \alpha R_k)^{d-k-1} s^2 C \binom{d}{k+1} (k+1)! 2^{dk - \binom{k+1}{2}}.$$

Suppose $k = d - t$, $1 \leq t \leq d - 2$, and $s_k = \lfloor 2^{(d-t)(t-1)/2-t} \sqrt{\alpha R_{k-1}} \rfloor$. Then

$$\begin{aligned} \frac{1}{Cd!} \left[f(\lambda_{s_k(s_k+1)/2}^k) - f(\lambda_1^k) \right] &\leq \frac{(s_k + \alpha R_k)^{t-1} s_k^2}{(t-1)!} 2^{\binom{d}{2} - \binom{t}{2} - 1} \\ &\leq \frac{(\alpha R_k)^{t-1} \alpha R_{k-1} 2^{(d-t)(t-1)-2t}}{(t-1)!} 2^{\binom{d}{2} - \binom{t}{2} - 1 + (t-1)} \\ &= \frac{\alpha^t (R_k 2^{d-t})^{t-1} t^{-1} R_{k-1}}{(t-1)!} 2^{\binom{d}{2} - \binom{t}{2} - t - 2} \\ &\leq \frac{\alpha^t R_{k-1}^t}{t!} 2^{\binom{d}{2} - \binom{t+1}{2} - 1}. \end{aligned}$$

Since we also have

$$\frac{1}{Cd!} \left[f(\lambda_{j+1}^{k-1}) - f(\lambda_j^{k-1}) \right] \geq \frac{(\alpha R_{k-1})^t}{t!} 2^{\binom{d}{2} - \binom{t+1}{2} - 1},$$

we may conclude that

$$f(\lambda_{j+1}^{k-1}) - f(\lambda_j^{k-1}) \geq f(\lambda_{s_k(s_k+1)/2}^k) - f(\lambda_1^k). \quad (6)$$

Let j_t , $1 \leq t \leq d - 2$, denote an integer so that $1 \leq j_t \leq s_t(s_t + 1)/2$, write $\mathbf{j} = (j_1, \dots, j_{d-2})$ and

$$\Lambda_{\mathbf{j}} = \lambda_{j_1}^{d-1} \cdot \lambda_{j_2}^{d-2} \cdot \dots \cdot \lambda_{j_{d-2}}^2,$$

where ‘ \cdot ’ is interpreted here as concatenation. By (6), $f(\Lambda_{\mathbf{j}})$ is a sequence of $\prod_{t=1}^{d-2} \frac{s_t(s_t+1)}{2}$ *distinct* integers. Furthermore, $\Lambda_{\mathbf{j}}$ is a partition of

$$\begin{aligned} n &= \sum_{t=1}^{d-2} 4^{t-1} (2s_t^2 + 8s_t + 1) + 2^{t-2} (\alpha R_{d-t} 2^{t+1} - 1) (2s_t + 1) \\ &\leq 3 \sum_{t=1}^{d-2} 4^t s_t^2 + 4^t \alpha R_{d-t} s_t \\ &\leq 3 \sum_{t=1}^{d-2} 4^t 2^{(d-t)(t-1)-2t} \alpha R_{d-t-1} + 4^t \alpha^{3/2} 2^{(d-t)(t-1)/2-t} R_{d-t} R_{d-t-1}^{1/2} \\ &\leq 3 \sum_{t=1}^{d-2} 2^{(d-t)(t-1)} \alpha R_{d-t-1} + \alpha^{3/2} 2^{(d-t)(t-1)/2+t} R_{d-t-1}^{3/2} \\ &\leq 6\alpha^{3/2} \sum_{t=1}^{d-2} 2^{(d-t)(t-1)/2+t} \left(8d^2 \alpha 2^{dt-t^2/2+t} \right)^{3/2} \\ &= 24\alpha^{3/2} d^3 \alpha^{3/2} \sum_{t=1}^{d-2} 2^{2dt-d/2-5t^2/4+2t} \end{aligned}$$

$$\langle (42d^4\alpha^{3/2}2^{3d^2/5+11d/10})\alpha^{3/2},$$

where the last inequality was obtained by maximizing the quadratic polynomial in the exponent of 2. On the other hand, the sequence Λ_j has length

$$\begin{aligned} 2^{-d+2} \prod_{t=1}^{d-2} s_t(s_t + 1) &\geq 2^{-d+2} \prod_{t=1}^{d-2} s_t^2 \\ &\geq 2^{-d+2} \prod_{t=1}^{d-2} 2^{(d-t)(t-1)-2t} \alpha R_{d-t-1} \\ &\geq (\alpha/2)^{d-2} \prod_{t=1}^{d-2} 2^{(d-t)(t-1)-2t} 10d^2 \alpha 2^{dt-t^2/2+t} \\ &\geq (5\alpha d^2 \alpha)^{d-2} \prod_{t=1}^{d-2} 2^{2dt-d-3t^2/2} \\ &= (5\alpha d^2 \alpha)^{d-2} 2^{\sum_{t=1}^{d-2} 2dt-d-3t^2/2} \\ &= (5\alpha d^2 \alpha)^{d-2} 2^{(d-2)(d^2-5d/2+5/2)} \\ &= (5d^2 \alpha 2^{d^2-5d/2+5/2})^{d-2} \alpha^{d-2} \\ &> n^{2(d-2)/3} 2^{-2d^3/5-1800}. \end{aligned}$$

□

Finally, we can prove Theorem 1.

Proof of Theorem 1. Applying Corollary 7, we have

$$\alpha \leq (k+3)^{2k+3} 2^{k^2/2+k+4}.$$

Therefore, the hypothesis of Theorem 11 gives the following lower bound on n :

$$\begin{aligned} 42k^4 \alpha^{3/2} 2^{3k^2/5+11k/10} &\leq 42k^4 (k+3)^{3k+9/2} 2^{3k^2/4+3k/2+6+3k^2/5+11k/10} \\ &\leq (k+3)^{3k+9} 2^{27k^2/20+13k/5+12}. \end{aligned}$$

Then, if $n \geq (k+3)^{3k+9} 2^{27k^2/20+13k/5+12}$, we may conclude that

$$\left| \{ \hat{f}_k(\lambda) : \lambda \vdash n \} \right| > 2^{-2d^3/5-1800} n^{2(d-2)/3}.$$

The number of parts in \mathcal{W}_j^k is at most

$$\begin{aligned} 2^{k-1} (2s+1) &\leq 2^{k-1} (2^{(k-t)(t-1)/2-t+1} \sqrt{\alpha R_{k-1}} + 1) \\ &\leq 2^{k-1} (2^{(k-t)(t-1)/2-t+1} \sqrt{\alpha^2 4e(k-1) 2^{2\binom{k}{2} - \binom{k+1}{2}} (d-k)!} + 1) \\ &\leq 2^{k-1} (2^{k^2/2} (k+3)^{2k+3} 2^{k^2/2+k+4} 5(k-1) 2^{-k/2} k^{k/2}) \end{aligned}$$

$$\leq (k+3)^{5k/2+4} 2^{k^2+3k/2+6}.$$

We may take $T = (k+3)^{5k/2+4} 2^{k^2+3k/2+6}$ to obtain a graph G_i with

$$(k+3)^{3k+9} 2^{27k^2/20+13k/5+11} + (k+3)^{5k/2+5} 2^{k^2+3k/2+6} < 2^{9k^2} =: N$$

vertices so that

$$\begin{aligned} \left| \{ \hat{f}_k(\lambda) : \lambda \vdash N \} \right| &> 2^{-2k^3/5-1800} n^{2(k-2)/3} \\ &= 2^{-1800} N^{-\frac{4}{81} \sqrt{\log_2 N} - \frac{4}{3}} \\ &= e^{\Omega((\log N)^{3/2})}. \end{aligned}$$

□

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