Problem Set 1 Solutions

MATH 777, Spring 2010, Cooper

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Each problem is worth 15 points. Full credit will be awarded for correct, rigorous solutions. Submitting your solutions in LaTeX gets you an automatic 5 bonus points. You are invited to work together on solutions, but please write up and submit your own proofs. Problem numbers in Diestel agree between the electronic and printed (3rd) editions, unless indicated otherwise.

1. We say that a graph G has an S^1-flow if it has a $\mathbb{C}$-circulation f so that $|f(\vec{e})| = 1$ for all $\vec{e} \in \vec{E}$. Show that, if G is cubic, it has an S^1-flow if and only if it has a 3-flow.

   Solution (Ada Park):

   Proof. ($\Rightarrow$) Suppose G has an S^1-flow. Then $|f(\vec{e})| = 1$ for all $\vec{e} \in \vec{E}$. In particular, for any fixed but arbitrary $\vec{e_0} \in \vec{E}$, we have that $|f(\vec{e_0})| = \mathrm{e}^{i\theta}$. Then let $f'(\vec{e}) = \mathrm{e}^{-i\theta} \cdot f(\vec{e})$. Then in particular, f' is an S^1 flow, and for our chosen $\vec{e}$, $f'(\vec{e}) = 1$.
   Then note that G is cubic, so if $\vec{e_0} = (e_0, x, y)$, then there exist $u, v \in V(G)$ such that $f'(\overrightarrow{xu}) + f'(\overrightarrow{xv}) = -1$. But this must mean then that

   $$\{f'(\overrightarrow{xu}), f'(\overrightarrow{xv})\} = \{\mathrm{e}^{\frac{2\pi i}{3}}, \mathrm{e}^{\frac{4\pi i}{3}}\}$$

   So in particular, in f, the edges incident to x must all differ by a multiple of $\mathrm{e}^{\frac{\pi i}{3}}$.
   Moreover, since our choice of $\vec{e}$ was arbitrary, we can repeat the process with each edge and find that all in f', all edges' flows are multiples of $\mathrm{e}^{\frac{\pi i}{3}}$ of their neighbors and hence by transitivity, of all other edges in the graph. So the values f' takes on are all in the set $\{\mathrm{e}^{\frac{k\pi i}{3}} | k = 0, 1, 2, 3, 4, 5\}$.
   But the only possible combinations for the edges leaving a vertex v are $\{\mathrm{e}^{\frac{k\pi i}{3}} | k = 0, 2, 4\}$ or $\{\mathrm{e}^{\frac{k\pi i}{3}} | k = 1, 3, 5\}$, and these choices are mutually

exclusive (since the elements of one set are the negative of the elements of the other set), and moreover, two vertices with outgoing edges from the same set cannot be adjacent, so this defines a bipartition on the vertices of G. But then G is cubic bipartite, which by a previous result implies that G has a 3-flow. ($\Leftarrow$) Suppose G is cubic and has a 3-flow. Then in particular, G is bipartite, with $V(G) = A \cup B$, with $A \cap B = \varnothing$. Every regular bipartite graph has a 1-factor, so choose any 1-factor M_1 of G. Then $G - M_1$ is even bipartite and so in particular contains a 1-factor also, so choose any 1-factor M_2 of $G - M_1$. Finally, $G - (M_1 \cup M_2)$ is 1-regular bipartite and so its edge set $(E(G - (M_1 \cup M_2)) := M_3)$ is also a 1-factor. In particular, every vertex of G is incident with exactly one edge from each of the three 1-factors M_1, M_2, M_3.
Then consider a function $f : \overrightarrow{E}(G) \to S^1$ that sends edges of $\{(e, a, b) | e \in M_1, a \in A, b \in B\}$ to 1, edges of $\{(e, a, b) | e \in M_2, a \in A, b \in B\}$ to $\mathrm{e}^{\frac{2\pi i}{3}}$, and edges of $\{(e, a, b) | e \in M_3, a \in A, b \in B\}$ to $\mathrm{e}^{\frac{4\pi i}{3}}$. In particular, f satisfies Kirchoff's law at every vertex of G. Then extend this function using (F1) to the directed edges (e, b, a) where $b \in B$ and $a \in A$. Then f still satisfies Kirchoff's law at every vertex and now also satisfies (F1), so it is an S^1 flow on G. □

2. Show that, if G has a 4-flow, then it contains a collection of cycles $\{C_1, \dots, C_N\}$ covering all of the edges (i.e., $\bigcup_{j=1}^N E(C_j) = E(G)$) so that
$$\sum_{j=1}^{N} \|C_j\| \le \frac{4}{3} \|G\|.$$

Solution (Virginia Johnson):

Proof:

Assume that G has a 4-flow. Then by Proposition 6.4.5, G is the union of two even subgraphs G_1 and G_2. Since the subgraphs G_1 and G_2 are even graphs, they are Eulerian (Theorem 1.8.1). Therefore, each subgraph is a union of disjoint cycles. Label this collection of cycles $\{C_1, \dots, C_N\}$, and note that this collection covers all edges of G. Note also that $|E(G_1) \backslash E(G_2)| + |E(G_1) \backslash E(G_2)| + |E(G_1 \bigcap G_2)| = \|G\|$. So $\bigcup_{j=1}^N E(C_j) = E(G) + E(G_1 \bigcap G_2))$. If it is always possible to find even subgraphs of G such that $|E(G_1 \bigcap G_2)| \le \frac{1}{3}\|G\|$, then $\sum_{j=1}^N \|C_j\| \le \|G\| + \frac{1}{3}\|G\|$ and the statement is proved. Since $|E(G_1) \backslash E(G_2)| + |E(G_1) \backslash E(G_2)| + |E(G_1 \bigcap G_2)| = \|G\|$, one of the terms must be less than $\frac{1}{3}\|G\|$. If $|E(G_1 \cap G_2)| \leq \frac{1}{3}\|G\|$, we are done, so assume wlog that $|E(G_1) \backslash E(G_2)| \leq \frac{1}{3}\|G\|$. Let G_3 be the graph with vertex set V whose edges are all those edges appearing in exactly one of G_1 and G_2. The degree of each vertex v in G_3 is the sum of the degrees of v in G_1 and G_2 minus twice the number of common edges adjacent to v and is therefore even. Then G_3 and G_1 are two even subgraphs of G. The intersection of the edge sets of the two even graphs is $E(G_1) \backslash E(G_2)$ and $|E(G_1) \backslash E(G_2)| \leq \frac{1}{3}\|G\|$. Therefore the statement is true.

3. Recall that, for $x \in \mathbb{R}$, a function $f : \overrightarrow{E} \to \mathbb{R}$ is an "x-flow" if it satisfies (F1) and (F2), and $1 \leq |f(\overrightarrow{e})| \leq x - 1$ for all $\overrightarrow{e} \in \overrightarrow{E}$. Show that, if G has an x-flow, then it has an (integral) $\lceil x \rceil$-flow.

   Solution (Bill Kay):

   We proceed by induction on the number of non-integral valued edges in G. If $k = 0$, then we have an $\lceil x \rceil$ flow and we are done. Now let $k \geq 1$, and let $e_0 = \{v_0 v_1\} \in E$ be an edge with non-integer flow value. Note that if a vertex v cannot be contained in only one edge with non-integral flow and satisfy Kirchhoff's law, and so we can find a vertex $e_1 = \{v_1 v_2\}$ with non integral flow. We proceed in this manner until we reach a vertex we have seen before, and then we have thus produced a cycle $C = \{e_0 e_1 \dots e_i e_0\}$ after an appropriate reindexing, such that each edge in the cycle has non-integer flow value. Now let $a := \min_{e \in C}\{\min\{|f(e)| - |\lfloor f(e) \rfloor|, |\lceil f(e) \rceil| - |f(e)|\}$. In otherwords, let a be the minimum distance to the nearest integer of the flow values on C. WLOG, say that $a = |f(e_0)| - |\lfloor f(e_0) \rfloor|$. Note that each edge $e \in C$ has some orientation $\overrightarrow{e}$ under f. We can define a new orientation g by the rule $g(e_0) = \overrightarrow{e_0}$, and the head of $\overrightarrow{e_i}$ meets the tail of $\overrightarrow{e_{i+1}}$. We say that g *agrees with* f *at* $e \in C$ if $g(e) = f(e)$. Otherwise, we say that g and f *disagree* at e. Now we can define a new flow h on G by the following rules:

   (a) the orientation of h agrees with f for each $e \in E$

   (b) $h(e) = f(e)$ for each $e \notin C$

   (c) If g agrees with f on C, then $h(e) = f(e) - a$. Otherwise, $h = f(e) + a$

   I claim that h is an x flow on G with one fewer non-integer valued edge. Clearly, for each $e \notin C$, $(F1)$ and $(F2)$ still hold. In fact, $(F1)$

still holds on C, since we are viewing the edges as undirected edges with the orientation dictating the sign. Now consider C. First, notice that $h(e_0)$ is integer valued, by the definition of a. We now must show that Kirchhoff's law still holds. Take $e_i, e_{i+1} \in C$ with common vertex v. Define $N_E(v) := \{e \in E : v \in e\}$, and $N_e'(v) := \{e \in E \setminus C : v \in e\}$. If $g(e_i)$ and $g(e_{i+1})$ agree with f, Then

$$\begin{aligned}\sum_{e \in N_E(v)} h(e) &= f(e_i) + a - (f(e_{i+1}) + a) \sum_{e \in N_E'(v)} f(v) \\ &= \sum_{e \in N_E(v)} f(v) \\ &= 0\end{aligned}$$

where the second flow value is being subtracted since it is leaving v. If $g(e_i)$ agrees with f, but $g(e_{i+1})$ disagrees, then

$$\begin{aligned}\sum_{e \in N_E(v)} h(e) &= f(e_i) + a + f(e_{i+1}) - a \sum_{e \in N_E'(v)} f(v) \\ &= \sum_{e \in N_E(v)} f(v) \\ &= 0\end{aligned}$$

Note that the other two cases follow by flipping the signs of the flow values of the first two cases. So Kirchhoff's law holds under h. All that we have to show now is that h is still a real valued x flow. Note that for each edge $e \notin C$, we have that $1 \leq |h(e)| \leq x - 1$ since h agrees with f on these edges. Moreover, if $e \in C$, then $h(e) - a \geq 1$ since $h(e) \geq 1$, and our choice of a was the minimum distance to the nearest integer on the cycle. Moreover, $h(e) + a \leq x - 1$ for the same reason. Since the only edges whose values were changed had non-integral flow, we have that if e had integeger flow value under f, then e has integer flow value under h. But then $h(e_0)$ is an integer, while $f(e_0)$ is not. By induction, there is an $\lceil x \rceil$ flow on G. Note that a similar argument holds if instead a was chosen as $|\lceil f(e) \rceil| - |f(e)|$ for some e, where we would be adding a where we subtracted (and vice versa) in this case. This completes the proof.

4. Determine $\varphi(Q_n)$ for $n \geq 1$, where Q_n is the n-dimensional hypercube, with vertex set $\{0,1\}^n$ and an edge xy whenever x and y differ in exactly one coordinate.

   Solution (Virginia Johnson):

   Proof: Q_1 does not have a flow, so $\varphi(Q_1) = \infty$.

   For $2n$, Q_{2n} is even and therefore has a 2-flow, so $\varphi(Q_{2n}) = 2$.

   Claim: For $2n+1$, $\varphi(Q_{2n+1}) = 3$.

   Since Q_{2n+1} is not even, it does not have a 2-flow. Since Q_{2n+1} is 4-edge connected, it has a 4-flow. Therefore, $\varphi(Q_{2n+1})$ is either 3 or 4. Note that Q_3 is a cubic bipartite graph and has a 3-flow (Proposition 6.4.2). Therefore, $\varphi(Q_3) = 3$. (The idea for the rest of this proof came from Austin). Since Q_n is the Cartesian product of n copies of K_2, and the Cartesian product is associative, $Q_{2n+1} = Q_{2n-2} \square Q_3$. Since Q_{2n-2} has a 2-flow and therefore a 3-flow and Q_3 has a 3-flow, if we can show that the Cartesian product of two graphs with a 3-flow has a 3-flow, then we are done.

   Claim: If G and H are graphs that admit a k-flow, then $G \square H$ admits a k-flow. Let g and h define a k-flow on G, and H respectively. For elements $u \in G, v \in H$ let $f(u,v)$ be defined as follows: For edges $\vec{e} = (e,(u,v),(u',v))$, let $f(\vec{e}) = g(e,u,u')$. For edges $\vec{e} = (e,(u,v),(u,v'))$, let $f(\vec{e}) = h(e,v,v')$. Using this construction it is clear that $f(\vec{e}) = -f(\overleftarrow{e})$ and that $0 < |f(\vec{e})| < k$ for all $\vec{e} \in \vec{E}(G \square H)$. Note that every $(e,(u,v),(u',v))$ edge has a corresponding edge $(e,u,u') \in \vec{E}(G)$, and similarly, $(e,(u,v),(u,v'))$ edge has a corresponding $(e,v,v') \in \vec{E}(H)$. Therefore, for any vertex $(u,v) \in G \square H$, $f((u,v),V(G \square H) = g(u,V(G)) + h(v,V(H)) = 0+0$, satisfying (F2). Since f satisfies the requirements for a k-flow, $G \square H$ admits a k-flow, satisfying (F2).

5. Diestel §6, #8

   Solution (Austin Mohr):

   Proof. Let f be given as above and let T be rooted at some vertex r. Consider a leaf v_0 (with corresponding pendant edge v_0w_0) at the lowest level of T. Let $V = V(G)$ and let $f(u,v)$ denote the flow from a vertex u to a vertex v. We have

   $$f(v_0,V) = \sum_{v \in N(v_0)} f(v_0,v)$$

$$= \sum_{\substack{v \in N(v_0) \\ v \neq w_0}} f(v_0, v) + f(v_0, w_0) \qquad \text{(since } H \text{ is abelian).}$$

Setting $f(v_0, w_0) = -\sum_{\substack{v \in N(v_0) \\ v \neq w_0}} f(v_0, v)$ gives $f(v_0, V) = 0$. To satisfy $f(v_0, w_0) = -f(w_0, v_0)$, we must of course set $f(w_0, v_0) = \sum_{\substack{v \in N(v_0) \\ v \neq w_0}} f(v_0, v)$.

Carry out this process levelwise (that is, assign flow to all edges of a given level, then proceed to the next level). More precisely, for each $v_k, w_k \in V$ with v_k below w_k in T, set $f(v_k, w_k) = -\sum_{\substack{v \in N(v_k) \\ v \neq w_k}} f(v_k, v)$ (since we assign flow levelwise, all the values in the summation will indeed by defined). Set also $f(w_k, v_k) = \sum_{\substack{v \in N(v_k) \\ v \neq w_k}} f(v_k, v)$. From this perspective, it is clear that the extension is unique and satisfies $f(v, w) = -f(w, v)$ for all $v, w \in V$, as there is a unique choice of flow at each $v \in V$ satisfying $f(v, V) = 0$. It is not evident, however, that $f(r, V) = 0$, as r is not below any vertex in T, and so no flow is chosen specifically to satisfy $f(r, V) = 0$. To verify that this constraint holds, observe that

$$\begin{aligned} 0 &= f(V, V) && \text{(since } f(\overrightarrow{e}) = -f(\overleftarrow{e}) \text{ for all } e \in G) \\ &= f(r, V) + f(V \setminus r, V) \\ &= f(r, V) && \text{(since } f(u, V) = 0 \text{ for all } u \neq r), \end{aligned}$$

and so f is indeed an H-circulation on G. □

6. Diestel §6, #10.

Solution (Travis Johnston):

Assume that a graph G has m spanning trees such that no edge $e \in E(G)$ appears in every spanning tree.

Claim: $\varphi(G) \leq 2^m$.

Proof. To show that $\varphi(G) \leq 2^m$ we will construct a $(\mathbb{Z}_2)^m$-flow on G. Let $T_1, \ldots, T_m$ be the m spanning trees. For $1 \leq i \leq m$ let $e_{i,1}, \ldots, e_{i,k_i}$ be the edges in $E(G) \setminus E(T_i)$. For each edge $e_{i,j}$ ($1 \leq i \leq m$ and $1 \leq j \leq k_i$) note that $T_i + e_{i,j}$ contains a unique cycle, $C_{i,j}$. Let $f_{i,j}(e) = (0, 0, \ldots, 1, 0, \ldots, 0)$ (the 1 in the i^{th} component) if $e \in E(C_{i,j})$, 0 otherwise. Note that $f_{i,j}$ satisfies (F1) trivially. Also, $f_{i,j}$ satisfies (F2) since it assigns a constant value around a single cycle and 0 elsewhere. Thus $f_{i,j}$ is a circulation for $1 \leq i \leq m$ and $1 \leq j \leq k_i$.

Let $f(e) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} f_{i,j}(e)$. It is clear that f is a circulation since it is the sum of circulations. We want to show that f is nowhere zero thereby concluding that f is a flow. To do this, we will show that if the i^{th} component of $f(e)$ is zero then $e \in E(T_i)$.

Fix an i such that $1 \le i \le m$. Suppose that the i^{th} component of $f(e)$ is zero and that $e \notin E(T_i)$. Then $e = e_{i,j}$ for some $1 \le j \le k_i$ and $e \in E(C_{i,j})$. Thus the i^{th} component of $f_{i,j}(e)$ is 1. Since the i^{th} component of $f_{\ell,k}(e)$ is zero for every $\ell \ne i$ and since (by assumption) the i^{th} component of $f(e)$ is zero, we have that $e_{i,j}$ must be in an even number of cycles $C_{i,k}$ for $1 \le k \le k_i$ (with fixed i). The only edges in $C_{i,k}$ are $e_{i,k}$ and edges in $E(T_i)$. Thus $e_{i,j}$ is in exactly one cycle, $C_{i,j}$ (with fixed i). This is a contradiction since $e_{i,j}$ must be in an even number of cycles. Thus we have that if the i^{th} component of $f(e)$ is zero, then $e \in T_i$. Then $f(e) = 0$ implies that $e \in T_i$ for $1 \le i \le m$ but this is contrary to our hypothesis that no edge is in every spanning tree. Therefore, f is a flow. □

7. Diestel §6, #12.

   Solution (Kamala Diefenthaler):

   Proof. Let C be a Hamilton cycle in $G = (V, E)$ and write C as $v_1 P_2 v_2 P_3 ... P_m v_m P_1 v_1$ where $\{v_i\}_{i=1}^{m}$ are the vertices with odd degree in G. Remember that the number of edges is equal to the sum of the degrees divided by 2. So the sum of the degrees are even, which implies that there are an even number of vertices of odd degree. Hence m is even. Now let $F = E(P_2) \cup E(P_4) \cup ... \cup E(P_m)$. Then define $\tilde{G} = (V, E \backslash F)$.
   Let $v \in V$. Case 1: $v = v_i$ for some $i = 1, ..., m$. Then $d_{\tilde{G}}(v) = d_G(v) - 1$. So $d_{\tilde{G}}(v)$ is even. Case 2: $v \in P_j$ for some j odd. Then $d_{\tilde{G}}(v) = d_G(v)$. So $d_{\tilde{G}}(v)$ is even. Case 3: $v \in P_k$ for some k even. Then $d_{\tilde{G}}(v) = d_G(v) - 2$. So $d_{\tilde{G}}(v)$ is even. Therefore $\tilde{G}$ is an even subgraph of G. Furthermore the construction of $\tilde{G}$ gives us that $G = \tilde{G} \cup C$. The union of two even graphs has a 4-flow. Thus G has a 4-flow.

   □

8. Diestel §6, #13.

   Solution (David Collins):

Since G has a 4-flow, we can create a $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow on G, f. Consider the sets of edges $A_1 = \{e \in E | f(e) = (1,0) or (1,1)\}, A_2 = \{e \in E | f(e) = (0,1) or (1,1)\}, A_3 = \{e \in E | f(e) = (1,0) or (0,1)\}$. Then the graph induced on A_1 must be an even graph, as it has a 2-flow induced by the first coordinate. Similarly, A_2 must also be even. Under an automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$, A_3 gets mapped to a similar set, so it must also be even. Thus the edges can be partitioned into 3 even graphs, with each edge showing up in 2. Thus it suffices to show that an even graph can be decomposed into cycles, with each edge in 1 cycle. We will prove this by induction on the number of edges in A_1. If A_1 has no edges, the statement is vacuously true. So suppose A_1 has edges. Since no vertex has degree 1, there are no leaves, so A_1 is not a forest, and thus has a cycle, C. Then $A_1 - C$ is a smaller even graph, and so by induction can be decomposed into cycles. Including C in this decomposition completes the decomposition of A_1. Thus the decompositions of each of the A_i's give a cycle double cover of G.