Problem Set 3 Solutions

MATH 776, Fall 2009, Cooper

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Each problem is worth 12 points. Full credit will be awarded for correct, rigorous solutions. Submitting your solutions in LaTeX gets you an automatic 5 bonus points. You are invited to work together on solutions, but please write up and submit your own proofs.

1. Diestel, §4, #13 (electronic & printed)

Solution: (Myself)

Suppose G is a straight-line plane graph drawn so that no edge is exactly horizontal or vertical, which we will represent as a set of points in $\mathbb{R}^2$ and a set of pairs of such points. We abuse notation somewhat by referring to the edges both as pairs of vertices and as point-sets of the corresponding line segments in $\mathbb{R}^2$. Enumerate $V(G)$ as $\{v_i = (x_i, y_i)\}_{i=1}^n$. Let $Z^2 = (\mathbb{Z} \times \mathbb{Z}, \{\{(a, b), (c, d)\} : |a - c| + |b - d| = 1\})$. Define the graph RG on vertex set $\{w_i = (Rx_i, Ry_i)\}_{i=1}^n$ with a straight-line segment edge between w_i and w_j iff $v_iv_j \in E(G)$. Finally, define the set S_R to be the subset of $\mathbb{Z} \times \mathbb{Z}$ which contains the four vertices (a, b), $(a, b + 1)$, $(a + 1, b + 1)$, and $(a + 1, b)$ precisely when the closed square $Q(a, b) = [a, a + 1] \times [b, b + 1]$ intersects RG, or $\|(x, y) - w_i\|_\infty \leq \sqrt{R}$ for some $i \in [n]$ and every $(x, y) \in Q(a, b)$.

Proposition 0.1. *$Z^2[S_R]$ (and therefore Z^2) contains G as a minor, for R sufficiently large.*

Proof. Let

$$X_i = \{v \in \mathbb{Z} \times \mathbb{Z} : \|v - (x_i, y_i)\|_\infty \leq \sqrt{R}\},$$

and, for $e \in E(RG)$, let

$$Y_e = \{v \in \mathbb{Z} \times \mathbb{Z} : d_\infty(v, e) \leq 1\}.$$

Claim 0.2. *$Z^2[X_i]$ is connected for each i.*

Proof. Suppose $(a, b), (c, d) \in X_i$. We may assume wlog (perhaps by reflection) that $a < c$ and $b < d$. Let $P_1 = \{a, a + 1, \ldots, c - 1, c\} \times \{b\}$ and $P_2 = \{c\} \times \{b, b + 1, \ldots, d - 1, d\}$. It is clear that $(a, b)P_1(c, b)P_2(c, d)$ is an $(a, b) - (c, d)$ path in Z^2; we need only show that $P_1 \cup P_2 \subset X_i$. This is immediate, however: since $|a| \leq \sqrt{R}$ and $|c| \leq \sqrt{R}$, it follows that $[a, c] \subset [Rx_i - \sqrt{R}, Rx_i + \sqrt{R}]$. Similarly, $[b, d] \subset [Ry_i - \sqrt{R}, Ry_i + \sqrt{R}]$. □

Claim 0.3. *$X_i \cap X_j = \emptyset$ for $i \neq j$.*

Proof. Suppose $(a, b) \in X_i$ and $(c, d) \in X_j$. Then

$$\begin{aligned} \|(a, b) - (c, d)\|_\infty &\geq \|w_i - w_j\|_\infty - \|(a, b) - w_i\|_\infty - \|(c, d) - w_j\|_\infty \\ &\geq R\|v_i - v_j\| - 2\sqrt{R}, \end{aligned}$$

by the triangle inequality. This last quantity is clearly positive for sufficiently large R. □

Claim 0.4. *$Z^2[Y_e]$ is connected for each $e \in E(G)$.*

Proof. Parameterize e by $r(t) = tw_i + (1-t)w_j$, $t \in [0,1]$. Again, we may assume $x_i < x_j$ and $y_i < y_j$. (The strictness arises from the assumption that no two adjacent vertices lie on a vertical or horizontal line.) For each t, define $S(t)$ to be the set consisting of those $v \in V(Z^2)$ so that $\|v - r(t)\|_\infty \leq 1$. Let T denote those $t \in [0,1]$ so that either coordinate of $r(t)$ is an integer. Since

$$T = (\pi_1 \circ r)^{-1}(\mathbb{Z} \cap [x_i, x_j]) \cup (\pi_2 \circ r)^{-1}(\mathbb{Z} \cap [y_i, y_j]),$$

where π_j is projection onto the j-th coordinate, T is finite. ($\pi_j \circ r$ is an invertible affine map and the arguments are finite sets.) Evidently, $S(t)$ is constant on each component of $[0,1] \setminus T$. Order these components left-to-right $C_1, \ldots, C_N$. Consider $t_1 \in C_i$ and $t_2 \in C_{i+1}$, for $i = 1, \ldots, N-1$. Then there is exactly one element t_0 of T that lies between t_1 and t_2. Let v be the (possibly nonunique) integer point closest in L^∞-norm to $r(t_0)$. By partitioning the unit square into four smaller squares, it is clear that we may assume that $\|v - r(t_0)\|_\infty \leq 1/2$. Therefore,

$$\begin{aligned} |v - r(t_j)|_\infty &\leq |v - r(t_0)|_\infty + |r(t_0) - r(t_j)|_\infty \\ &\leq \frac{1}{2} + \epsilon < 1, \end{aligned}$$

since we may take $t_1 + \epsilon = t_0 = t_2 - \epsilon$ for some sufficiently small $\epsilon > 0$. For both $j = 1, 2$, then, $|v - r(t_j)|_\infty < 1$, so $v \in S(t_1) \cap S(t_2)$. Therefore, $S(C_i) \cap S(C_j) \neq \emptyset$, so, for $1 \leq \ell \leq N-1$,

$$Y_e^\ell \subseteq \bigcup_{i=1}^{\ell} S(C_i)$$

induces a subgraph of Z^2 which intersects $S(C_{\ell+1})$. In particular, $Y_e = Y_e^N$ induces a connected subgraph of Z^2.

□

Claim 0.5. *$Y_e \cap X_i \neq \emptyset$, $Y_e \cap X_j \neq \emptyset$, $Y_e \setminus \{X_i \cup X_j\} \neq \emptyset$, and $Y_e \cap X_k = \emptyset$ for each $e = \{w_i, w_j\} \in E(RG)$ and $w_k \notin e$.*

Proof. The first two claims are essentially the same and follow immediately from the fact that $v_i \in Y_e$. As for the third claim, let v be the vertex of Z^2 L^∞-closest to the midpoint m of e. Then

$$\begin{aligned} \|v - w_i\|_\infty &\geq \|m - w_i\|_\infty - \|v - m\|_\infty \\ &\geq \frac{R}{2}\|w_i - w_j\|_\infty - 1 \\ &> \sqrt{R} \end{aligned}$$

for sufficiently large R. Thus, while $v \in Y_e$, we have $v \notin X_i$, and by symmetry, $v \notin X_j$. Therefore, $v \in Y_e \setminus \{X_i \cup X_j\}$. For the last claim, let z be any point of Y_e. Then

$$\begin{aligned} \|z - w_k\|_\infty &\geq d_\infty(e, w_k) - d_\infty(e, z) \\ &\geq R d_\infty(\{v_i, v_j\}, v_k) - 1, \end{aligned}$$

which is clearly positive for sufficiently large R, since v_k does not lie on the line segment whose endpoints are v_i and v_j. □

We may now finish the proof of the proposition. Consider a new graph Γ obtained from $Z^2[S_R]$ as follows. First, for each edge $e = v_i v_j \in E(RG)$, all of $Z^2[Y_e \setminus \bigcup_i X_i]$ except an X_i-X_j path P_e is removed from $Z^2[S_R]$; the existence of such a path is guaranteed by the connectivity of $Z^2[Y_e]$ and the fact that Y_e intersects both X_i and X_j. That P_e is nonempty follows from the fact that $Y_e \setminus \{X_i \cup X_j\} \neq \emptyset$ and the fact that $X_i \cap X_j = \emptyset$. Next, each X_i is contracted to a single vertex, something which is possible because we know that $Z^2[X_i]$ is connected. Since $Y_e \cap X_k = \emptyset$ for each $v_k \notin e$, it follows that Γ is simply a subdivision of G. □

Proposition 0.6. *Any minor of a planar graph is planar.*

Proof. Of course, this follows immediately from Kuratowski's Theorem. However, we are forbidden to use that tool. So, we take a direct approach. It is clear that removing edges of a graph retains planarity; we need only show that contraction of an edge does so as well. So, suppose G is a plane graph drawn with straight-line segments, and $e = xy \in E(G)$. Since no vertex v or edge of G intersects e except for x, y, and e itself, there is some small $\epsilon > 0$ so that the open set $X = \{w \in \mathbb{R}^2 : d(e, w) < \epsilon\}$ intersects $V(G)$ precisely in $\{x, y\}$ and intersects $E(G)$ only in a subset of $E_G(x) \cup E_G(y)$. Since X consists of a rectangle with a half-disk glued to each end, it is convex. Therefore, $X \cap G$ contains e and a half-open line segment ℓ_f for each $f \in E_G(x) \cup E_G(y) \setminus \{e\}$. Let m be the midpoint of e, and define ℓ'_f to be the closed line segment connecting $f \cap \partial X$ to m for each $f \in E_G(x) \cup E_G(y) \setminus \{e\}$. Define a new graph G' with vertex set $V(G) \setminus \{x, y\} \cup \{m\}$ and edge set

$$E(G) \setminus \{E_G(x) \cup E_G(y)\} \cup \{f \setminus X \cup \ell'_f : f \in E_G(x) \cup E_G(y) \setminus \{e\}\}.$$

Evidently, G' is a drawing of G/e, where each vertex except x and y has retained its location, and the new (contracted) vertex is m. Since

$$X \cap G = \bigcup_{f \in E_G(x) \cup E_G(y) \setminus \{e\}} \ell_f,$$

there are no new crossings introduced by the modification of G on the set X. Furthermore, because the line segments ℓ'_f meet at m, they do not cross anywhere else. □

Theorem 0.7. *A graph is planar iff it is the minor of a grid.*

Proof. This is immediate from the two propositions above: any planar graph is a minor of a grid, and any minor of a grid (which is clearly planar) is itself planar. □

2. Diestel, §4, #18 (electronic & printed)

Solution: (Myself)

Proof. Suppose G is a maximal plane graph, and xy an edge to be added to G. Since G is 3-connected, there are 3 independent $x - y$ paths P_1, P_2, and P_3 in G, by Menger's Theorem. Furthermore, each P_i has length at least 2, since $xy \notin E(G)$. Let C denote the graph induced by the neighbors of x; by the proof of Theorem 5.4.2, C is a cycle, and x is the only vertex of G lying in the face of C which contains x. By the Jordan Curve Theorem, then, each P_i intersects C, since x and y lie in different faces of C. For each i, $i = 1, 2, 3$, let z_i denote the *last* vertex of C occurring on P_i, and let P'_i denote the $z_i - z_{i+1}$ path (indices interpreted modulo 3) contained in C which does not contain z_{i+2}. The paths xz_1, xz_2, xz_3, xy, P'_1, P'_2, P'_3, z_1P_1y, z_2P_2y, and z_3P_3y are independent, by construction. Along with the vertices x, y, z_1, z_2, and z_3, they form a TK^5 in $G + xy$.

Now, suppose x has degree at least 4 in G. Let w be any element of $C \setminus \{z_1, z_2, z_3\}$. Suppose (wlog) that $w \in P'_1$. Let $S_1 = \{w, z_3, y\}$ and $S_2 = \{z_1, z_2, x\}$. Then S_1 and S_2 form the two color classes of a $TK_{3,3}$ in $G + xy$, with edges wP'_1z_1, wP'_1z_2, wx, P'_2, P'_3, z_3P_3y, yP_1z_1, yP_2z_2, and xy. Similarly, if y has degree at least 4 in G, then $G + xy$ contains a $TK_{3,3}$. Therefore, we may assume that x and y both have degree exactly 3. Let $N_G(y) = \{w_1, w_2, w_3\}$. If $N_G(y) \neq N_G(z)$, then we may assume wlog that P_1 contains z_1 and w_1, where $z_1 \neq w_1$. Let $S_1 = \{x, z_3, w_1\}$ and $S_2 = \{z_1, z_2, y\}$. Then S_1 and S_2 form the two color classes of a $TK_{3,3}$ in $G + xy$ with edges xz_1, xz_2, xy, P'_3, P'_1, z_3P_3y, $w_1P_1z_1$, w_1y, and $w_1Qw_2P_2z_2$, where Q is the $w_1 - w_2$ path contained in $G[N_G(y)]$ that avoids w_3. Finally, suppose $N_G(x) = N_G(y)$. Then the vertices $\{x, y, z_1, z_2, z_3\}$ induce a $\Gamma = \overline{K_2} * K_3$. It is easy to see that Γ is 3-connected, so it has only one planar embedding (up to homeomorphism of S^2; see Figure 1).

Note that each face of this drawing is incident to either x or y. Since we know that $|G| \geq 6$, there is at least one more vertex of G; call it w. Clearly, w lies in one of the faces f of Γ. Furthermore, since G is 3-connected, there are 3 independent $w-v_i$ paths Q_i, where $\{v_1, v_2, v_3\}$ are the vertices of f. In particular, we may assume that Q_i contains v_i and no other v_j for $j \neq i$, for each $i = 1, 2, 3$. We may also assume that $v_1 = x$. (The case of $v_1 = y$ is identical.) Note that the penultimate vertex u of Q_1 must be interior to the cycle $C = v_1v_2v_3v_1$. Indeed, if not, then either $u = v_2$, $u = v_3$, or, by the Jordan Curve Theorem, the path wQ_iu intersects C; in each case, Q_i contains two vertices of C, contradicting the independence of the Q_i's. Then the edge ux is not uz_1, uz_2, or uz_3, so x has degree greater than 3, a contradiction. □

3. Diestel, §4, #20 (electronic & printed)

Solution: (Austin Mohr)

Definition 0.8. *A graph is called outerplanar if it has a drawing in which every vertex lies on the boundary of the outer face.*

Proposition 0.9. *A graph is outerplanar if and only if it contains neither K^4 nor $K_{2,3}$ as a minor.*

Proof. (⇒) Let G be outerplanar. Define the graph G' by placing a new vertex v in the unbounded face of G and connecting v to every vertex of G. Since G is outerplanar, these edges can be added without inducing a crossing, and so G' is planar. By Kuratowski's Theorem, G' contains no K^5 nor $K_{3,3}$ minor. Hence, it must be that G contains no K^4 nor $K_{2,3}$ minor (else adding the vertex v and the corresponding edges would result in a K^5 or $K_{3,3}$ minor in G').

(⇐) Let G contain no K^4 nor $K_{2,3}$ minor. Construct the graph G' as before. Evidently, G' contains no K^5 nor $K_{3,3}$ (else G would contain a K^4 or $K_{2,3}$ minor), and so G' is planar. A priori, we do not know where v is located in a planar drawing of G'. To address this, project G' to the sphere S^2. Using an appropriate homeomorphism from S^2 to S^2 and then projecting back to the plane, we can obtain a planar drawing of G' in which v lies in the unbounded face of G. The fact that v can reach every vertex of G without inducing a crossing implies that every vertex of G is on the boundary of the unbounded face. That is, G is outerplanar. □

4. Diestel, §4, #23 (electronic & printed)

Solution: (Travis Johnston)

Proof. Let G be a 2-connected plane graph. Since G is 2-connected every face is bounded by a cycle. If G is bipartite then G contains no odd cycles. Therefore if G is bipartite then every face is bounded by an even cycle.

Now, suppose that every face of G is bounded by an even cycle. Say that $C_1, C_2, ..., C_f$ are the cycles that bound the faces. It is clear that every induced cycle of G is a facial boundary and that for $1 \leq i \leq f$ C_i is an induced cycle. Thus by proposition 1.9.1, $C_1, ..., C_f$ generate $\mathcal{C}(G)$. We want to show that any cycle in G has an even number of edges, equivalently and even number of vertices. Let C be any cycle in G and write

$$C = a_1C_1 + a_2C_2 + ... + a_fC_f$$

where $a_i \in \{0, 1\}$. By only considering the cycles in the sum with non-zero coefficients we may assume (WLOG) that $C = C_1 + C_2 + ... + C_k$. We can rename the cycles $C_1, ..., C_k$ in such a way that $(C_1 + C_2 + ... + C_i) \bigcap C_{i+1} \neq \emptyset$ for $1 \leq i \leq k-1$. If we could not order the cycles in this way then $C_1 + C_2 + ... + C_k$ would form several disjoint cycles, but $C = C_1 + C_2 + ... + C_k$ and C is a single cycle. Finally, we want to show that $C_1 + ... + C_i$ is an even cycle for $2 \leq i \leq k$

thus establishing that C is an even cycle. Now, we think of each of the C_i's as a column vector in $\mathbb{F}_2$. Suppose that C_a and C_b are any two even cycles (boundaries of faces) with non-empty (edge) intersection. The column vector C_a has $|C_a|$ (an even number) of 1's, likewise, C_b has $|C_b|$ (an even number) of 1's. Say that m is the number of edges that C_a and C_b have in common. Then $C_a + C_b$ has $|C_a| + |C_b| - 2m$ 1's which is also an even number. Thus $C_a + C_b$ is an even cycle. We now apply this argument to our cycles $C_1, ..., C_k$. We have that $C_1 + C_2$ is an even cycle. Then $(C_1 + C_2) + C_3$ is an even cycle. By continuing in this manner, assuming that $C_1 + ... + C_i$ is an even cycle, we have that $(C_1 + ... + C_i) + C_{i+1}$ is an even cycle. Thus we conclude that $C = (C_1 + C_2 + ... + C_{k-1}) + C_k$ is an even cycle. Thus every cycle in G is even. Therefore, G is bipartite. □

5. Diestel, §5, #11 (electronic & printed)

Solution: (Scott Dunn)

Proof. Let G be a graph. Claim: The graph G is critical 3-chromatic if and only if G is an odd cycle.

(⇒) Suppose that G is an odd cycle. Then G is 3-chromatic. Removing any vertex $v \in V(G)$ leaves a path, which is 2-chromatic. Thus G is a critical 3-chromatic graph.

(⇐) Suppose that G is critical 3-chromatic. Since G is 3-chromatic, G cannot be bipartite. By Proposition 1.6.1, G contains an odd cycle C. By way of contradiction, suppose that G is more than just the odd cycle C. Take a vertex $v \in G \setminus C$. But $G - \{v\}$ still contains C, so $G - \{v\}$ is 3-chromatic. This is a contradiction since we assumed that G is critical 3-chromatic. Therefore G is just the odd cycle C.

Therefore a graph G is critical 3-chromatic if and only if G is an odd cycle. □

6. Diestel, §5, #13 (electronic & printed)

Solution: (Tatiana Orlova)

Given $k \in \mathbb{N}$, find a constant $c_k > 0$ such that every large enough graph G with $\alpha(G) \leq k$ contains a cycle of length at least $c_k|G|$.

Proof.

Suppose that G is t-chromatic, i.e. $\chi(G) = t$. Then $\alpha(G)$ is a size of a largest set of vertices in G that can be colored by the same color, so that $\frac{|G|}{t} \leq \alpha(G) \leq k$. Thus we have $\frac{1}{k}|G| \leq t$. By corollary 5.2.3 in Diestel G has a subgraph G_t, such that $\delta(G_t) \geq t - 1$. We know that the size of G is sufficiently large so we can claim that $|G| \geq 3k$. Then $\frac{1}{k}|G| \leq t$ implies that $t \geq 3$. Since $\delta(G_t) \geq t - 1 = 2$ by proposition 1.3.1 in Diestel G_t has a cycle C_{G_t} such that $|C_{G_t}| \geq \delta(G_t) + 1 = t \geq \frac{1}{k}|G|$. Since $C_{G_t} \subseteq G_t \subseteq G$ setting up $c_k = \frac{1}{k}$ will guarantee that G must have a cycle of length at least $c_k|G|$.♣

7. Diestel, §5, #23 (electronic & printed)

Solution: (Travis Johnston)

Algorithm Let G_1 be an isolated vertex. Let G_2 be a K_2. For $k > 2$ construct G_k from G_{k-1} in the following manner. Label the vertices of G_{k-1} as $v_{(1,1)}, v_{(1,2)}, ..., v_{(1,n)}$. Then, let the vertex set of G_k be $\{v_{(1,1)}, ...v_{(1,n)}, v_{(2,1)}, ..., v_{(2,n)}, v_\star\}$. Also, let the edge set of G_k be the union of the following three sets:

$$\{\{v_{(1,i)}, v_{(1,j)}\} : v_{(1,i)} \sim v_{(1,j)} \text{ in } G_{k-1}\}$$

$$\{\{v_{(2,i)}, v_{(1,j)}\} : v_{(1,i)} \sim v_{(1,j)} \text{ in } G_{k-1}\}$$

$$\{\{v_\star, v_{(2,i)}\} : 1 \le i \le n\}.$$

Claim: The graph G_k is triangle free and $\chi(G_k) = k$.

Proof. It is clear that G_1 is triangle-free and $\chi(G_1) = 1$. It is also clear that G_2 is triangle-free and $\chi(G_2) = 2$. First we will show that if G_{k-1} is triangle-free then G_k is also triangle-free for $k > 2$. Then we will show that $\chi(G_k) = \chi(G_{k-1}) + 1$. We will then conclude that G_k is triangle-free for $k \ge 1$ and that $\chi(G_k) = k$.

Suppose that $k > 2$ and that G_{k-1} is triangle-free. If $v_\star$ were in a triangle it would be of the form: $v_\star, v_{(2,i)}, v_{(2,j)}, v_\star$. However, this is impossible since $v_{(2,i)}$ is not adjacent to $v_{(2,j)}$ for any $1 \le i, j \le n$. By the same reasoning, there is no triangle of the form $v_{(2,i)}, v_{(1,m)}, v_{(2,j)}, v_{(2,i)}$. There are also no triangles of the form $v_{(1,i)}, v_{(1,j)}, v_{(1,m)}, v_{(1,i)}$ since G_{k-1} is triangle-free and $G_k[v_{(1,1)}, v_{(1,2)}, ..., v_{(1,n)}] \cong G_{k-1}$. The only remaining possibility for a triangle is one of the form $v_{(2,i)}, v_{(1,j)}, v_{(1,m)}, v_{(2,i)}$. By the construction, $v_{(2,i)} \sim v_{(1,j)}$ if and only if $v_{(1,i)} \sim v_{(1,j)}$. Likewise, $v_{(2,i)} \sim v_{(1,m)}$ if and only if $v_{(1,i)} \sim v_{(1,m)}$. Then $v_{(2,i)}, v_{(1,j)}, v_{(1,m)}, v_{(2,i)}$ is a triangle in G_k if and only if $v_{(1,i)}, v_{(1,j)}, v_{(1,m)}, v_{(1,i)}$ is a triangle. By the previous remark we know the latter cannot be a triangle in G_k. Thus we have shown that G_k is triangle-free.

Now we will show that $\chi(G_k) = \chi(G_{k-1}) + 1$. We can $\chi(G_{k-1}) + 1$ color G_k in the following manner. Take a valid $\chi(G_{k-1})$ coloring on $G_k[v_{(1,1)}, ..., v_{(1,n)}] \cong G_{k-1}$ and color vertex $v_{(2,i)}$ with the same color as vertex $v_{(1,i)}$. Then color $v_\star$ with a new color. This is a valid coloring since the only neighbors of $v_{(2,i)}$ are $v_\star$ and neighbors of $v_{(1,i)}$ in $G_k[v_{(1,1)}, ..., v_{(1,n)}]$ and $v_\star$ is colored differently than every other vertex in G_k. Thus $\chi(G_k) \le \chi(G_{k-1}) + 1$. Clearly $\chi(G_k) \ge \chi(G_{k-1})$ since G_k contains a copy of G_{k-1}. Suppose that G_k can be colored with $\chi(G_{k-1})$ colors. Then every color must be used at least once to color the vertices $v_{(1,1)}, ..., v_{(1,n)}$. If every color were also used at least once to color $v_{(2,1)}, ..., v_{(2,n)}$ then it would be impossible to color $v_\star$ without using a new color. Thus there must be at least one color missing in the set $\{v_{(2,1)}, ..., v_{(2,n)}\}$. WLOG suppose that $v_{(1,1)}$ is colored with a color not used on the vertices $v_{(2,1)}, ..., v_{(2,n)}$. Let $V_1 := \{v_{(1,1)}, ..., v_{(1,m)}\}$ be the set of all vertices in $\{v_{(1,1)}, ..., v_{(1,n)}\}$ that have the same color as $v_{(1,1)}$. And let $V_2 := \{v_{(2,i)} : v_{1,i} \in V_1\}$. Since no two vertices in V_1 are adjacent then $G_k[(\{v_{(1,1)}, ..., v_{(1,n)}\} \setminus V_1) \bigcup V_2] \cong G_{k-1}$. Now we have a contradiction since $G_k[(\{v_{(1,1)}, ..., v_{(1,n)}\} \setminus V_1) \bigcup V_2]$ is colored with strictly fewer than $\chi(G_{k-1})$ colors and is isomorphic to G_{k-1}. Thus $\chi(G_k) > \chi(G_{k-1})$ which implies that $\chi(G_k) = \chi(G_{k-1}) + 1$. □

8. Suppose $G = (V, E)$ and $G' = (V', E')$. On the vertex set $V \times V'$, let $G \square H$ denote the Cartesian/direct product : $(a, b) \sim (c, d)$ iff $((a = c) \wedge (b \sim d)) \vee ((b = d) \wedge (a \sim c))$; let $G \times H$ denote the categorical/tensor product : $(a, b) \sim (c, d)$ iff $(a \sim c) \wedge (b \sim d)$; let $G \boxtimes H$ denote the strong/normal product : $(a, b) \sim (c, d)$ iff $((a = c) \vee (a \sim c)) \wedge ((b = d) \vee (b \sim d))$; and let $G[H]$ denote the lexicographic/replacement product : $(a, b) \sim (c, d)$ iff $(a \sim c) \vee ((a = c) \wedge (b \sim d))$. Show that:

1. $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$
2. $\chi(G \times H) \le \min\{\chi(G), \chi(H)\}$
3. $\max\{\chi(G), \chi(H)\} \le \chi(G \boxtimes H) \le \chi(G[H]) \le \chi(G)\chi(H)$

Solution: (Kamala Diefenthaler)

Proof. 1. $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$

Note that G and H are subgraphs of $G \square H$. Thus $\chi(G \square H) \ge \max\{\chi(G), \chi(H)\}$. WLOG suppose $\max\{\chi(G), \chi(H)\} = \chi(G) = n$. Let $c_0 : G \to \{0, 1, ..., n-1\}$ be an n coloring of G. Then define $c_i(v) = (c_0(v) + i) \mod n$ for $i = 1...n-1$ and for all $v \in G$. Also let d be an $\chi(H) \le n$ coloring of H. Let $c : V(G \square H) \to \{1, ..., n\}$ defined by $c((g, h)) = c_{d(h)}(g)$. Suppose (x, y) and (w, z) are adjacent in $G \square H$. Then either $(x = w) \wedge (y \sim z)$ or

$(y = z) \wedge (x \sim w)$.
If $(x = w) \wedge (y \sim z)$, then

$$d(y) \neq d(z) \Rightarrow c_{d(y)}(x) \neq c_{d(z)}(x) = c_{d(z)}(w) \Rightarrow c((x, y)) \neq c((w, z))$$

If $(y = z) \wedge (x \sim w)$, then

$$c_{d(y)}(x) \neq c_{d(y)}(w) = c_{d(z)}(w) \Rightarrow c((x, y)) \neq c((w, z))$$

Thus c is a proper coloring of $G \square H$. Hence $\chi(G \square H) \leq \max\{\chi(G), \chi(H)\}$. Therefore $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$.

2. $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$
WLOG suppose $\min\{\chi(G), \chi(H)\} = \chi(G) = n$. Let d be an n coloring of G. Let $c : V(G \square H) \to \{1, ..., n\}$ defined by $c((g, h)) = d(g)$. Suppose (x, y) and (w, z) are adjacent in $G \square H$. Then $(x \sim w) \wedge (y \sim z)$. So $d(x) \neq d(w) \Rightarrow c((x, y)) \neq c((w, z))$. Thus c is a proper coloring of $G \times H$. Therefore $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$.

3. $\max\{\chi(G), \chi(H)\} \leq \chi(G \boxtimes H) \leq \chi(G[H]) \leq \chi(G)\chi(H)$
By definition, $G \square H$ is a subgraph of $G \boxtimes H$ which is a subgraph of $G[H]$. Then we have that $\max\{\chi(G), \chi(H)\} = \chi(G \square H) \leq \chi(G \boxtimes H) \leq \chi(G[H])$.
It remains to show that $\chi(G[H]) \leq \chi(G)\chi(H)$. Let $c_1 : V(G) \to S_1$ be a $\chi(G)$ coloring of G, and let $c_2 : V(H) \to S_2$ be a $\chi(H)$ coloring of H. Let $c : V(G \square H) \to S_1 \times S_2$ defined by $c((g, h)) = (c_1(g), c_2(h))$. Suppose (x, y) and (w, z) are adjacent in $G[H]$. Then $(x \sim w) \vee ((x = v) \wedge (y \sim z))$. If $(x \sim w)$, then

$$c_1(x) \neq c_1(w) \Rightarrow (c_1(x), c_2(y)) \neq (c_1(w), c_2(z)) \Rightarrow c((x, y)) \neq c((w, z))$$

If $(x = v) \wedge (y \sim z)$, then

$$c_2(y) \neq c_2(z) \Rightarrow (c_1(x), c_2(y)) \neq (c_1(w), c_2(z)) \Rightarrow c((x, y)) \neq c((w, z))$$

Thus c is a proper coloring of $G[H]$. Therefore $\chi(G[H]) \leq |S_1 \times S_2| = \chi(G)\chi(H)$.

□

9. The *thickness* $\Theta(G)$ of a graph G is the minimum k so that $G = \bigcup_{i=1}^{k} G_i$ for some planar graphs G_i, $1 \leq i \leq k$. Show that any graph G with $\|G\| > 0$ satisfies

$$\left\lceil \frac{\delta(G)}{6} \right\rceil \leq \Theta(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil.$$

(Hint on the upper bound: expand G to a $2\lceil\frac{\Delta(G)}{2}\rceil$-regular graph.

Solution: (David Collins)

Proof. Suppose G has n vertices. Then $n\delta(G) \leq 2m$ where m is the number of edges, so $m \geq \frac{n\delta(G)}{2}$. Then if we split G into k planar graphs, then the edges of G have been split into k sets. So by the pigeon-hole principle, one of the sets has at least $\frac{n\delta(G)}{2k}$ edges. But each of these graphs is planar, and a planar graph has at most $3n - 6$ edges. Thus $\frac{n\delta(G)}{2k} \leq 3n - 6 < 3n$, so $k \geq \frac{\delta(G)}{6}$. Since k is an integer, this gives $\left\lceil \frac{\delta(G)}{6} \right\rceil \leq k$

For the other inequality, we define a sequence of graphs G_i with $G_0 = G$. To construct G_i for $i > 1$, we take $G_{i-1} \square K^2$, which increases the degree of every vertex by 1, and if the degree of a vertex a is equal to $2\left\lceil \frac{\Delta(G)}{2} \right\rceil$ in G_{i-1}, remove the new edge added to it in $G_{i-1} \square K^2$. So the maximum degree of G_i is $2\left\lceil\frac{\Delta(G)}{2}\right\rceil$, since no edges are added to vertices with that degree, and all vertices start with degree less than $\Delta(G)$. Furthermore, since G_{i-1} is a subgraph of G_i, $\Theta(G_i) \geq \Theta(G_{i-1})$. Also, since we are adding an edge to every vertex of small enough degree, the minimum degree is rising as i increases. But the degree can't get any higher than $2\left\lceil\frac{\Delta(G)}{2}\right\rceil$, so once the minimum degree equals that, all of the vertices must have that degree, since it can't be higher or lower. Thus for some k, G_k is $2\left\lceil\frac{\Delta(G)}{2}\right\rceil$-regular. Then G_k is regular of even degree, so by Corollary 2.1.5, G_k has a 2-factor, call it H_0. Then H_0 is planar, and every vertex has degree 2, so $G_k - H_0$ is regular of degree $2\left\lceil\frac{\Delta(G)}{2}\right\rceil - 2$, so we can continue pulling out 2-factors until all of the edges are used. Since each 2-factor is planar, and every edge is in one of the 2-factors, we have split G_k into planar subgraphs. Since each vertex has degree 2 in each 2-factor, and no edge is used twice, there are $\left\lceil\frac{\Delta(G)}{2}\right\rceil$ 2-factors. Thus $\left\lceil\frac{\Delta(G)}{2}\right\rceil \geq \Theta(G_k) \geq \Theta(G_0)$. □

10. Prove that a finite graph is chordal and outerplanar iff it is a disjoint union of K_4-minor free near-triangulations. (A *chordal* graph has the property that any cycle longer than a C_3 induces a chord. A *near-triangulation* is a connected graph with a planar embedding so that all inner faces are triangles.) Be rigorous!

Solution: (Myself)

Proof. Evidently, we may assume that the graph G in question is connected.

Claim 0.10. *If G is chordal and outerplanar, then it is K_4-minor free and a near-triangulation.*

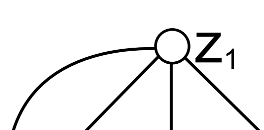
Proof. That G is K_4-minor free follows immediately from a previous problem (above). Note that, if every block of G is a near-triangulation, then G is. Indeed, this follows from a simple induction on the size of the block graph H of G. It is clearly true if $H = \emptyset$. Then either the block graph has a leaf, in which case G can be written as $G = G_1 \cup G_2$ where $|G_1 \cap G_2| = 1$ and both G_i are outerplanar, or else G has a leaf v and $G - v$ is outerplanar. If the blocks of $G - v$ are a near-triangulation, then $G - v$ is a near-triangulation by the inductive hypothesis. Since $G - v$ is outerplanar, the vertex x to which v is adjacent in G lies on the outer face. Therefore, given any drawing of $G - v$, one can append a short line segment to x which lies in the outer face to obtain a drawing of G. Then no face except the outer face is affected, so G is a near-triangulation. In the former case, suppose that G can be written as $G_1 \cup G_2$, where G_1 and G_2 intersect in a single vertex x. Since G_1 is outerplanar, it has a drawing in which x lies on its outer face. The outer face contains an open circular segment whose boundary contains x, since two edges incident to x lie on the outer face, forming a positive angle. G_2 can be drawn in the closure of this region (say, by contracting both coordinates sufficiently and then rotating) so that the embedding of G_2 intersects that of G_1 precisely in $\{x\}$.
Therefore, we need only show that the inner faces of the blocks of a 2-connected G are triangles. Suppose not; then some inner face f has more than three vertices. However, G is chordal, so two vertices v and w of f which are not adjacent in $G[f]$ are indeed adjacent in G. The edge vw cannot be drawn in f because it would separate f in $\mathbb{R}^2 \setminus G$. Likewise, it cannot be drawn in the outer face f' of $G[f]$ (which is a cycle by 2-connectedness, so that $\mathbb{R}^2 = f \cup G[f] \cup f'$). We then have a contradiction, and may conclude that G is a near-triangulation. □

Claim 0.11. *If G is K_4-minor free and a near-triangulation, then G is chordal and outerplanar.*

Proof. Outerplanarity follows immediately from the definition of near-triangulation. We claim that G is 3-connected. Let G_u be G plus all edges of the form uv, where u is some fixed vertex on the outer face and v is any non-neighbor of u on the other face. Since a plane graph is maximal plane iff all of its faces are triangles, G_u is maximal plane for any u, which implies that it is 3-connected. Suppose there were some vertex set $\{x, y\}$ of G so that $G - \{x, y\}$ were disconnected. Then G_x is also disconnected by $\{x, y\}$, a contradiction.

Suppose G is not chordal, so some cycle C of length at least 4 has no chords. $G - C$ cannot be empty, since C by itself is not a near-triangulation. Therefore, G contains some vertex $u \notin C$. Since G is 3-connected, there are 3 independent $u - C$ paths uQ_1t_1, uQ_2t_2, and uQ_3t_3. Then the vertices u, t_1, t_2, and t_3, along with the Q_i's and the three paths into which t_1, t_2, and t_3 separate C, are a K_4 minor, a contradiction. □

□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