Problem Set 1 Solutions

MATH 776, Fall 2009, Cooper

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1. Prove Theorem 1.5.1 in Diestel.

Solution (David Collins):

Proof. i→ii: Suppose for contradiction that we have two distinct xy paths in T, call them P_1 and P_2. Each of these paths must contain at least one edge which the other does not, so let e_1 be the first edge in P_1 which is not in P_2. Let a_1 be the vertex preceding e_1 in P_1, thus a_1 is in both P_1 and P_2. Let a_2 be the next vertex in P_2 which is also in P_1, and occurs after a_1 in P_1. Note that such a vertex must exist, because if no other vertex fills the requirement, y does. Then P_1 can be restricted to a path from a_1 to a_2, call it Q_1. Similarly, P_2 can be restricted to a path from a_1 to a_2, call it Q_2. By definition, Q_1 and Q_2 have only their endpoints in common. Thus Q_1 followed by Q_2 is a cycle. But T is a tree and thus has no cycles, so this is a contradiction.
ii→iii: Clearly T is connected since there is a path between any 2 vertices. Choose an edge e. Let x and y be the endpoints of e. Then is an xy path in T, so it must be the unique xy path in T. Thus $T - e$ does not have an xy path, so it is disconnected. Therefore T is minimally connected.
iii→iv: If T contains a cycle, then we may remove any edge from the cycle without disconnecting the graph, so T does not contain any cycles. Choose vertices x and y such that xy is not an edge in T. Since T is connected, there is an xy path in T. Then adding the edge xy creates a cycle. Thus T is maximally acyclic.
iv→i: By assumption T is acyclic, so we need only show that it is connected. Choose any vertices x and y in T. If the edge xy is in T, then clearly there is an xy path. So suppose xy is not in T. Then adding xy creates a cycle C in $T + xy$. Then $C - xy$ is in T, and is an xy path in T. Thus T is connected, and therefore a tree. ☐

2. For graphs G and H, G is called "uniquely H-saturated" if it does not contain H as a subgraph, but $G + e$ contains a single (unlabeled) copy of H for each $e \in \binom{V(G)}{2} \setminus E(G)$. Prove that, if G is uniquely K_3-saturated, then G is either a star or one of a finite list of graphs.

Solution (Bill Kay):

Note 1: If G is disconnected, $G+e$ will not contain a triangle if e goes from one component to another, and so we can assume that G is connected. First, suppose G is a tree. Notice that a single edge is a star, and is uniquely H saturated. Suppose that there are at least 2 vertices in G. Note that for any two non-adjacent vertices, the distance is at most 2(i.e. G is diameter 2), for if $d(u, v) \geq 3$, addition of the edge uv would not induce a K_3. .We can be assured that such a pair exists, because adding an edge forms a triangle, and so there must have been a P_2 between the vertices of the edge added to G. So pick a pair of non-adjacent vertices, and call the P_2 between them uvw. If this is the entire graph, it is indeed a star, and we are done. Suppose that there is a vertex x not on this path. By connectivity, there must exist a ux path. If this path is not through v, then there is some vertex x_1 adjacent to u which is not through v. But uvw is the unique path from w to u, and ux_0 is the unique path from u to x_0, so the distance from w to x_0 is 3, which cannot happen. So the ux path has length at most 2, and it must go through v. This gives that G must have been a star, since every vertex which is not v must be adjacent to v. Now suppose G contains a cycle. The smallest induced cycle cannot be size 3 by definition of K_3 saturated. The shortest induced cycle cannot be of length 4, because adding in either of the missing edges gives 2 triangles, contradicting the uniqueness. Notice that any edge added to a 5 cycle satisfies the definition. So a graph with girth 5 is a candidate. Suppose G has girth 6. Let $C \in G$ be a length 6 cycle. Choose $u, v \in C$ such that $d_C(u, v) = 3$. It is clear that such antipodal vertices exist. G has diameter $2 \Rightarrow d_G(u, v) = 2$. So there exists a w such that $u \sim w$, $w \sim v$. Notice that $d_C(u, v) = 3 \Rightarrow w \notin C$. Take uvw with $uPv \in C$ where uPv is length 3. This yields a 5 cycle, contradicting the supposition that G had girth 6. This argument carries through for any graph with girth larger than 6 by simply taking vertices on the cycle which achieve the diameter of a cycle. So all the candidates of graphs which contain cycles have girth 5. Notice that the diameter 2 condition still holds. And so diameter 2 girth 5 graphs are the only candidates for uniquely K_3 saturated graphs, and there are only at most 5 of them (Stated in class). This completes the proof.

3. Diestel §1, problem 2.

Solution (Austin Mohr):

Definition 0.1. Let $d \in \mathbb{N}$ and $V = \{0, 1\}^d$; thus, V is the set of all 0-1 sequences of length d. The graph on V in which two such sequences form an edge if and only if they differ in exactly one position is called the d-dimensional cube (denoted Q_d).

Proposition 0.2. The average degree of the d-dimensional cube is d. (In fact, the d-dimensional cube is d-regular.)

Proof. Consider a vertex $v \in \{0, 1\}^d$. As v is a binary string of length d, there are precisely d other binary strings (also of length d) that differ from v in exactly one position. By the above definition, v is adjacent to each of the vertices represented by these binary strings, and so v has degree d. Hence, the d-dimensional cube is d-regular, and so has average degree d. □

Proposition 0.3. The number of edges of the d-dimensional cube is $d \cdot 2^{d-1}$.

Proof. We make use of the handshaking lemma, i.e. for any graph G

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v)$$

By problem 3a, this becomes

$$\begin{aligned} |E(Q_d)| &= \frac{1}{2} \sum_{v \in \{0,1\}^d} d(v) \\ &= \frac{1}{2} \cdot |\{0,1\}^d| \cdot d \\ &= \frac{1}{2} \cdot 2^d \cdot d \\ &= d \cdot 2^{d-1} \end{aligned}$$

□

Proposition 0.4. The diameter of the d-dimensional cube is d.

Proof. We first claim that $diam(Q_d) \leq d$. To see this, let $v, w \in \{0, 1\}^d$. Define the path $P = v_0 v_1 \ldots v_d$, where $v_0 = v$ and

$$v_{i+1} = \begin{cases} v_i & \text{if } v_i \text{ and } w \text{ agree at position } i+1 \\ u & \text{otherwise, where } u \text{ is the unique vertex that differs from } v_i \text{ only at position } i+1 \end{cases}$$

(We will ignore multiple occurrences of a vertex in this path.) By the definition of Q_d, we are assured that P is connected (i.e. we can always reach the vertex u). Furthermore, P is acylic, as the construction can never return to a previously visited vertex. Finally, the construction gives $v_d = w$. Hence, P is indeed a $v - w$ path. Now, in the worst case, we will add an edge at each step, giving a path of length d. Hence, the distance between v and w is at most d, i.e. $diam(Q_d) \leq d$.

In the above algorithm, let $v = 0^d$ and $w = 1^d$. Since v and w differ at every bit, we will have to add an edge at each step, and so the upper bound is realized. Therefore, $diam(Q_d) = d$. ☐

Proposition 0.5. The girth of the d-dimensional cube is 4 for $d \geq 2$.

Proof. We first claim that the girth of Q_d is at least 4. Suppose, to the contrary, that we can find a cycle of length 3. Let u, v, and w be the (distinct) vertices of this cycle. Since $u \sim v$, u and v differ at exactly one bit (say, bit i). Similarly, $v \sim w$ means that v and w differ at exactly one bit (say, bit j). Now, it cannot be that $i = j$, or else $u = w$. So, u and w differ at two bits, yet $u \sim w$, which is a contradiction. Hence, the girth of Q_d is at least 4.

Now, let x be any binary string of length $d - 2$. Observe that $00x \sim 10x \sim 11x \sim 01x \sim 00x$ is a cycle of length 4 in Q_d. Therefore, the girth of Q_d is exactly 4. ☐

Proposition 0.6. The circumference of the d-dimensional cube is 2^d for $d \geq 2$.

Proof. (by induction on d)
Basis: $Q_2 = C_4$, and so has a cycle of length $4 = 2^2$.
Induction: Suppose Q_k has a cycle of length 2^k. Partition $V(Q_{k+1})$ into sets V_1 and V_2, where

$$\begin{aligned} V_1 &= \{v \in V(G) \mid v \text{ begins with a 0 bit}\} \\ V_2 &= \{v \in V(G) \mid v \text{ begins with a 1 bit}\} \end{aligned}$$

Observe that the induced subgraph on V_1 is isomorphic to Q_k (define the isomorphism to simply delete the first bit of a vertex), and so there is a cycle on length 2^k on V_1. Similarly, we can find a cycle of length 2^k on V_2. Without loss of generality, let it be that the edge between 0^d and 010^{d-2} belongs to the cycle on V_1 (some isomorphic labeling of the vertices of V_1 can ensure this). Similarly, require that the edge between 10^{d-1} and 110^{d-2} belongs to the cycle on V_2. In addition to these two cycles, include the edge between 0^d and 10^{d-1} and the edge between 010^{d-2} and 110^{d-2}. The result is a 4-cycle on these vertices. Hence, the removal of the edge between 0^d and 010^{d-2} and the edge between 10^{d-1} and 110^{d-2} forms a single cycle of length 2^d, as desired. ☐

4. Diestel §1, problem 3.

Solution (Kamala Diefenthaler):

Proof:

Let G be a graph containing a cycle C, and assume that G contains a path P of length at least k between two vertices of C.

Suppose that $|C| \geq \sqrt{k}$. Then we are done.

So let $|C| < \sqrt{k}$. Then the maximum number of times that P can intersect C is at every vertex of C. So $|P \cap C| < \sqrt{k}$. Let $P \cap C = \{v_0, v_1, ..., v_n\}$, and let P_i be a subset of P starting from v_{i-1} and ending at v_i $(0 - 1 = n)$. Define

$$m = \max_{1 \leq i \leq n} (|P_i|)$$

Then there exists a j such that $|P_j| = m$. So we have that

$$k \leq |P| = \sum_{i=1}^{n} |P_i| \leq \sum_{i=1}^{n} m \leq nm < \sqrt{k}m$$

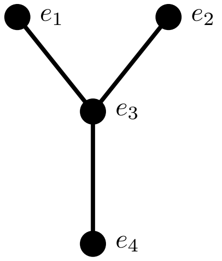
Thus $m > \sqrt{k} \Rightarrow |P_j| > \sqrt{k}$.

Start at v_{j-1} and follow P_j until you reach v_j, then take C back to v_{j-1}. This is a cycle in G, and the length is greater than $\sqrt{k}+1$.

5. Prove that a line graph is claw-free (i.e., no induced $K_{1,3}$'s).

Solution (Travis Johnston):

Proof. Suppose that $L(G)$ contains a claw. Consider the subgraph of the line graph induced on the claw. The graph below is the resulting induced subgraph. Note that the vertices in $L(G)$ are labelled by their corresponding edges in the graph G.



Since the above graph is the induced subgraph of $L(G)$ on $\{e_1, e_2, e_3, e_4\}$ we know that every edge between these 4 vertices in $L(G)$ is still present in the induced subgraph. Since $e_1 \sim e_3$ in $L(G)$ we know that edges e_1 and e_3 share an endpoint in G. Likewise, since $e_2 \sim e_3$ in $L(G)$ we know that edges e_2 and e_3 share an endpoint in G. Similarly, $e_4 \sim e_3$ so e_3 and e_4 share an endpoint in G. Since e_3 only has 2 endpoints in G at least two of e_1, e_2, and e_4 must share an endpoint. But if this were the case then the two vertices would be adjacent in $L(G)$ and they are not. Thus line graphs cannot contain a claw as an induced subgraph. ☐

6. Prove that any regular connected bipartite graph is 2-connected.

Solution (Austin Mohr):

Remark 0.7. The following problem is modified slightly to consider only k-regular graphs for $k \geq 2$. The only connected, 1-regular graph is a single edge, and Diestel's definition of k-connectedness insists that $|G| > k$.

Proposition 0.8. For $k \geq 2$, any k-regular, connected, bipartite graph is 2-connected.

Proof. Let $k \geq 2$, and let G be a k-regular, connected, bipartite graph on partite sets A and B. Suppose, for the purpose of contradiction, that G possesses a cutvertex v. Without loss of generality, assume $v \in A$. Let C be some component of $G - v$. We count $|E(C)|$ in two ways.

Consider the vertices of $A \cap V(C)$. As $v \in A$, v is not adjacent to any vertex in $A \cap V(C)$. Hence, each vertex in $V(C)$ has degree k in C, and so k divides $|E(C)|$.

On the other hand, consider the vertices of $B \cap V(C)$. As v is a cutvertex for G, v is adjacent to at least one vertex in $B \cap V(C)$. Let m be the number of vertices of $B \cap V(C)$ that are adjacent to v. Observe that, if all k of v's neighbors belonged to $B \cap V(C)$, removing v would not disconnect the graph. Hence, $m < k$. Now, we have that m of the vertices in $B \cap V(C)$ have degree $k-1$ in C and the remaining vertices have degree k in C. It follows that

$$\begin{aligned} |E(C)| &= m(k-1) + (|B \cap V(C)| - m)k \\ &= mk - m + |B \cap V(C)|k - mk \\ &= |B \cap V(C)|k - m \end{aligned}$$

which is not divisible by k, contradicting the previous claim that $|E(C)|$ is divisible by k.

Therefore, G does not possess a cutvertex, and so G is 2-connected. ☐

7. Diestel §1, problem 19.

Solution (Austin Mohr):

Definition 0.9. Let G be a connected graph, and let $r \in G$ be a vertex. Starting from r, move along the edges of G, going whenever possible to a vertex not visited so far. If there is no such vertex, go back along the edge by which the current vertex was first reached (unless the current vertex is r; then stop). This procedure is known as depth-first search.

Proposition 0.10. The edges traversed during a depth-first search form a normal spanning tree in G with root r.

Proof. Let G be a connected graph and let T be a subgraph of G resulting from the depth-first search procedure beginning at some vertex r in G. Observe that, since G is connected, T spans G. Moreover, since the procedure never visits the same vertex twice (save for the backtracking step, during which no edge is introduced), T is acyclic. Hence, T is a spanning tree of G.

Now, let u and v be a pair of adjacent vertices in G. Without loss of generality, let it be that u was visited first by the depth-first search procedure. To show that T is in fact normal in G, it suffices to show that vertices u and v are comparable in the tree order of T (since T is a spanning tree of G). When the depth-first search procedure reaches the vertex u (either the first time or during one of the subsequent backtracking steps), we will have v adjacent to u in G and v unvisited by the procedure, so the edge between u and v will be added to T. Hence, when the procedure terminates, u will lie on the unique rv-path in T. In other words, $u \preceq v$ in the tree order of T. Therefore, T is a normal spanning tree in G. ☐

8. Diestel §1, problem 21.

Solution (Kamala Diefenthaler):

Proof:

Let T be a tree. For $|T| = 1$ the result is trivial.

Suppose $|T| = 2$. Then there are two automorphisms of T.

(1) The identity map which fixes everything.

(2) A map that switches the two vertices. Then the edge is fixed.

Now suppose that for all $|T| \leq n$, every automorphism of T fixes a vertex or an edge.

Let $|T| = n+1$. Then let $v_1, v_2, ..., v_k \in V(T)$ be the vertices of degree one in T. There is at least one v_i as T is a tree. Now let ϕ be an automorphism of T. Then $\phi(v_i) \in \{v_1, v_2, ..., v_k\}$ for all $1 \leq i \leq k$, as automorphisms preserve degrees. Consider $H = T - \{v_1, v_2, ..., v_k\}$. Thus $\phi|_H$ is an automorphism of H. Futhermore, H is a non-empty tree and $|H| < n+1$. So by the induction hypothesis, $\phi|_H$ fixes a vertex or an edge of H. Therefore ϕ fixes a vertex or an edge of $|T|$.

9. Diestel §1, problem 33.

Solution (Travis Johnston):

Proof. Suppose that G has n vertices and m edges. Let G_i be the i'th connected component and say that G_i has m_i edges and n_i vertices for $1 \leq i \leq k$. We know that $\sum_{i=1}^k m_i = m$ and $\sum_{i=1}^k n_i = n$. By Theorem 1.9.6 we know that $\dim \mathcal{C}(G_i) = m_i - n_i + 1$ and $\dim \mathcal{C}^*(G_i) = n_i - 1$. Since the connected components are disjoint, their cycles are linearly independent. Then

$$\dim \mathcal{C}(G) = \sum_{i=1}^{k} \dim \mathcal{C}(G_i) = \sum_{i=1}^{k} (m_i - n_i + 1) = m - n + k$$

and

$$\dim \mathcal{C}^*(G) = \sum_{i=1}^{k} \dim \mathcal{C}^*(G_i) = \sum_{i=1}^{k} n_i - 1 = n - k.$$

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10. Show that the only finite connected graph that is isomorphic to its own line graph is a cycle.

Solution (David Collins):

Proof. Let G be a graph, and let G' be its line graph. Since the number of vertices in G' is equal to the number of edges in G, if $G = G'$ then G has the same number of vertices as it does edges. Then if T is a spanning tree of G, T has one less edge than G. Then by adding this edge back to T, we create a unique cycle. Thus G has exactly one cycle. If e is an edge on the cycle of G, then in the line graph, it is a vertex which is connected to the vertices of its two neighboring edges in the cycle. So the vertex in G' which comes from e is also on a cycle. Furthermore, since the number of edges in a cycle is the same as the number of vertices, each vertex which is on the cycle in G' comes from an edge on the cycle in G. Suppose for contradiction that there is a vertex in G which is not on the cycle. Since G is connected, there is a path from this vertex to any vertex in the cycle. Consider the shortest of these paths. Then none of the edges in the path are in the cycle, and the final vertex is in the cycle, and has degree at least three, call this vertex y. Consider the last edge in this path, call it x. Let a and b be the two edges connected to y that are in the cycle. Then x goes to a vertex in G' which is connected to the two vertices in G' which come from a and b. But these vertices are connected to each other because they are both incident at y. This creates a cycle. Then x gets taken to a point on a cycle, but x is not on a cycle. This is a contradiction. Thus G has no points not on the cycle, so G is a cycle. □