Final Exam Solutions

MATH 776, Fall 2009, Cooper

December 25, 2009

Each problem is worth 15 points. Full credit will be awarded for correct, rigorous solutions. Submitting your solutions in LaTeX gets you an automatic 5 bonus points. You may *not* work together on the solutions, although you may ask *me* about the problems. Also, you may assume without proof anything proven in class or stated in one of the problem sets, or that could reasonably be considered prerequisite for the course. However, you may not use the Weak or Strong Perfect Graph Theorems anywhere. Finally, keep in mind that Diestel uses the convention that P^k has k edges.

1. The family $\mathcal{C}$ of *cographs* consists of (1) K^1, (2) the complement of any cograph, (3) the disjoint union of any two cographs. Prove (a) that the cographs are precisely those graphs which contain no induced P^3 and (b) cographs are perfect.

Solution: (Myself)

Proposition 0.1. *G is a cograph iff it contains no induced P^3.*

Proof. First, suppose that G is the smallest cograph that contains an induced P^3. Since $G \neq K^1$, G is either the disjoint union of two smaller cographs, or else is the complement of the disjoint union of two smaller cographs. In the first case, it is clear that G contains an induced P^3 iff one of its components does. However, its components are smaller than G, and are therefore induced P^3-free. In the second case, since $P^3 \simeq \overline{P^3}$, $\overline{G}$ also contains an induced P^3. Again, this means that one of the components of $\overline{G}$ induces a P^3, a contradiction.

Now, suppose that G is induced P^3-free. We wish to show that G is a cograph. We proceed by induction. The base case $|G| = 1$ is trivial. Suppose, then, that every induced P^3-free graph on fewer than $|G|$ vertices is a cograph. Choose any vertex $v \in V(G)$, and let $G' = G - v$. Since G' is induced P^3-free, it is a cograph by induction. Therefore, either $G' = K^1$, or G' is the disjoint union of two cographs, or G' is the join of two cographs. In the first case, it is clear that G is then a cograph. The third case can be transformed into the second case by complementation. Therefore, G' is the disjoint union of two cographs G_1 and G_2. Suppose v has no edges to G_1. Then $G = G_1 \cup G_2^*$, where $G_2^* = G[V(G_2) \cup \{v\}]$. Since G_1 and G_2^* are induced P_3-free and smaller than G, they are cographs. We may then conclude that G is a cograph. Similarly, if v has no edges to G_2, then G is a cograph. Now, suppose v is adjacent to every vertex in G_1. Then $\overline{G} = \overline{G_1} \cup \overline{G_2^*}$. Since $\overline{G_1}$ and $\overline{G_2^*}$ are induced P_3-free (indeed, P_3 is self-complementary and G_2^* is induced P_3-free) and smaller than G, the inductive hypothesis applies and we may conclude that $\overline{G}$ is a cograph. Then G is a cograph. Similarly, if v is adjacent to every vertex of G_2, then G is a cograph. Putting the pieces together, we may assume that v is adjacent to some vertex of G_1 and to some vertex of G_2, and nonadjacent to some vertex of G_1 and to some vertex of G_2.

Let $G_{11} = G_1[N(v)]$, $G_{12} = G_1 - G_{11}$, $G_{21} = G_2[N(v)]$, and $G_{22} = G_2 - G_{21}$. The above argument implies that all four of these subgraphs of G' are nonempty, and the inductive hypothesis implies that they are all cographs. Suppose $a \in V(G_{11})$, $b \in V(G_{12})$, $c \in V(G_{21})$, and $d \in V(G_{22})$. If $ab \in G_1$, then $bavc$ is an induced P^3 in G, a contradiction. Similarly, $cd \notin G_2$. Therefore, $G_1 = G_{11} \cup G_{12}$ and $G_2 = G_{21} \cup G_{22}$. But then we may write $G = G_3 \cup G_4$, where $G_3 = v * (G_{11} \cup G_{21})$ and $G_4 = G_{12} \cup G_{22}$. Clearly, G_3 and G_4 are cographs, so G is as well. □

Proposition 0.2. *Every cograph is perfect.*

Proof. We proceed by induction. The case of $|G| = 1$ is trivial. Suppose that every cograph on fewer than $|G| > 1$ vertices is perfect. Note that part (a) of this problem implies that any induced subgraph of a cograph is a cograph; then, any proper induced subgraph of G is a cograph, and by the inductive hypothesis, perfect. Therefore, to show that G is perfect, it suffices to show that $\omega(G) \leq \chi(G)$.

We may write $G = G_1 \cup G_2$ or $G = G_1 * G_2$, where G_1 and G_2 are cographs. Since G_1 and G_2 are strictly smaller than G, the inductive hypothesis implies that they are perfect. In the first case,

$$\begin{aligned}\omega(G) &= \max\{\omega(G_1), \omega(G_2)\}\\ &\leq \max\{\chi(G_1), \chi(G_2)\}.\end{aligned}$$

However, $\chi(G)$ is at most $\max\{\chi(G_1), \chi(G_2)\}$, since the same set of colors may be used to color G_1 and G_2. Thus, $\omega(G) \leq \chi(G)$. In the second case,

$$\begin{aligned}\omega(G) &= \omega(G_1) + \omega(G_2)\\ &\leq \chi(G_1) + \chi(G_2).\end{aligned}$$

Now, $\chi(G)$ is at most $\chi(G_1)+\chi(G_2)$, since we may color G_1 and G_2 with disjoint sets of colors to obtain a coloring of G. Thus, $\omega \leq \chi(G)$. ☐

2. Find a closed form for the chromatic polynomial of the wheel graph $K^1 * C_n$ and $K_{2,s}$.

Solution: (Austin Mohr)

Lemma 0.3. *The chromatic polynomial of the cycle on n vertices is $(\lambda-1)^n + (-1)^n(\lambda-1)$.*

Proof. (by induction on n)

For $n = 3$, observe that C_3 is the same as K^3, giving $\lambda(\lambda-1)(\lambda-2) = \lambda^3 - 3\lambda^2 + 2\lambda$ colorings. Comparing this with the proposed formula, we see

$$\begin{aligned}p_{C_3}(\lambda) &= (\lambda-1)^3 + (-1)^3(\lambda-1)\\ &= (\lambda^3 - 3\lambda^2 + 3\lambda - 1) - (\lambda - 1)\\ &= \lambda^3 - 3\lambda^2 + 2\lambda,\end{aligned}$$

and so the formula hold in the case of $n = 3$.

For $n \geq 4$, observe that

$$\begin{aligned}p_{C_n}(\lambda) &= p_{C_n - e}(\lambda) - p_{C_n/e}(\lambda) && (\text{some } e \in E(C_n))\\ &= p_{P_{n-1}}(\lambda) - p_{C_{n-1}}(\lambda)\\ &= \lambda(\lambda-1)^{n-1} - \left[(\lambda-1)^{n-1} + (-1)^{n-1}(\lambda-1)\right]\\ &= \lambda(\lambda-1)^{n-1} - (\lambda-1)^{n-1} - (-1)^{n-1}(\lambda-1)\\ &= \lambda(\lambda-1)^{n-1} - (\lambda-1)^{n-1} + (-1)^{n}(\lambda-1)\\ &= (\lambda-1)^{n-1}(\lambda-1) + (-1)^{n}(\lambda-1)\\ &= (\lambda-1)^{n} + (-1)^{n}(\lambda-1),\end{aligned}$$

as desired. ☐

Proposition 0.4. *The chromatic polynomial of the wheel graph $K^1 * C_n$ is*

$$\lambda\left[(\lambda-2)^{n-1} + (-1)^{n-1}(\lambda-2)\right].$$

Proof. Let v be the central vertex of the wheel graph W_n. First, choose any of the available λ colors to color v. Since v is adjacent to all other vertices, the chosen color cannot be used again. Hence, it remains to color $W_n - v$ (i.e. C_n) using $\lambda - 1$ colors. Making use of 0.3, we have

$$\begin{aligned} p_{W_n}(\lambda) &= \lambda p_{C_n}(\lambda - 1) \\ &= \lambda\left[(\lambda-2)^n + (-1)^n(\lambda - 2)\right], \end{aligned}$$

as desired. □

Proposition 0.5. *The chromatic polynomial of* $K_{2,s}$ *is* $\lambda(\lambda-1)\left[(\lambda-1)^{s-1} + (\lambda-2)^s\right]$.

Proof. Let v and w be the vertices in the partite set of size 2. We consider two cases.

If v and w are colored the same, then we choose a single color of the available λ colors. As both v and w are adjacent to all of the remaining s vertices, we color using the remaining $\lambda - 1$ colors. As these s vertices are independent, we can color freely, and so arrive at $\lambda(\lambda-1)^s$ colorings for this case.

If v and w are colored differently, then there are $\lambda(\lambda - 1)$ ways to color them. As before, we use the remaining $\lambda - 2$ colors to color the remaining s vertices freely, giving $\lambda(\lambda-1)(\lambda-2)^s$ colorings for this case.

Taken together, we have that

$$\begin{aligned} p_{K_{2,s}}(\lambda) &= \lambda(\lambda-1)^s + \lambda(\lambda-1)(\lambda-2)^s \\ &= \lambda(\lambda-1)\left[(\lambda-1)^{s-1} + (\lambda-2)^s\right], \end{aligned}$$

as desired. □

3. An *interval* representation of a graph $G = (V, W)$ is a family $\mathcal{I}$ of intervals of the real line and a bijection $\phi : V \to \mathcal{I}$ so that $vw \in E$ iff $\phi(v) \cap \phi(w) \neq \emptyset$. Such a representation is "proper" if no element of $\mathcal{I}$ contains another. Prove that an interval graph is claw-free (no induced $K_{1,3}$) iff it has a proper representation.

Solution: (Myself)

Proof. First of all, it is easy to see that we may assume that the $\phi(v)$'s are all closed intervals. Suppose G contains an induced $K_{1,3}$ with bipartition $\{a\}$ and $\{b, c, d\}$. Let $\phi(a) = [a_1, a_2]$, $\phi(b) = [b_1, b_2]$, $\phi(c) = [c_1, c_2]$, and $\phi(d) = [d_1, d_2]$, with $b_1 \leq c_1 \leq d_1$. Since $\phi(b) \cap \phi(c) = \emptyset$, $b_1 < b_2 < c_1 < c_2$. Likewise, since $\phi(c) \cap \phi(d) = \emptyset$, $c_1 < c_2 < d_1 < d_2$. Note that $d_1 > a_1$, since otherwise $b_2 < d_1 \leq a_1$ implies that $\phi(b) \cap \phi(a) = [b_1, b_2] \cap [a_1, a_2] = \emptyset$. But then $d_1 \leq a_2$, since otherwise $\phi(d) \cap \phi(a) = [d_1, d_2] \cap [a_1, a_2] = \emptyset$. Similarly, $b_2 \geq a_1$. However, since $b_2 < c_1 < c_2 < d_1$, this implies that $c_1 > a_1$ and $c_2 < a_2$, i.e., $\phi(c) \subset \phi(a)$. So any representation of G is improper.

Now, suppose that G is claw-free. Let ϕ be the representation so that the number N of pairs $a, b \in V(G)$ so that $\phi(a) \subset \phi(b)$ is minimized. Note that the intervals $\phi(v)$, $v \in V(G)$, are a poset under inclusion, so we may choose a minimal element x. Then, for all $y \in V(G) \setminus \{x\}$, $\phi(y) \not\subset \phi(x)$, but there exists some $z \in V(G)$ so that $\phi(x) \subset \phi(z)$. (If no such x exists, then $N = 0$ and we are done, since then G has a proper representation.) Let $\phi(x) = [x_1, x_2]$ and $\phi(z) = [z_1, z_2]$, so $z_1 \leq x_1 < x_2 \leq z_2$. There is some $\epsilon > 0$ so that $[z_2, z_2 + \epsilon]$ contains no endpoint of any $\phi(v)$, $v \in V(G) \setminus \{z\}$. Define a new function ϕ' of $V(G)$ so that $\phi'(v) = \phi(v)$ for all $v \in V(G) \setminus \{x\}$, and $\phi'(x) = [x_1, z_2 + \epsilon]$.

Clearly, $\phi(v) \cap \phi(w) \neq \emptyset$ iff $\phi'(v) \cap \phi'(w) \neq \emptyset$ whenever $x \notin \{v, w\}$. Since $\phi'(x) \supset \phi(x)$, if $\phi(v) \cap \phi(x) \neq \emptyset$, then $\phi'(v) \cap \phi'(x) \neq \emptyset$. Suppose that $\phi(v) \cap \phi(x) = \emptyset$. Let $\phi(v) = [v_1, v_2]$. If $v_2 < x_1$, then $\phi'(v) \cap \phi'(x) = \emptyset$. So, in order for ϕ' *not* to be a representation of G, we must have $v_1 > x_2$. If $v_1 > z_2$, then $v_1 > z_2 + \epsilon$, so $\phi(v) \cap \phi(x) = \emptyset$. Therefore, if no $\phi(u)$, $u \in V(G)\backslash\{x, z\}$, has its left endpoint in $[x_2, z_2]$, then ϕ' is a representation of G. Furthermore, there exists no u so that $\phi(u) \not\subset \phi(x)$ but $\phi'(u) \subset \phi'(x)$, since then the left endpoint of $\phi(u)$ would have to lie in $[x_2, z_2]$. Since now $\phi'(x) \not\subset \phi'(z)$, the number of interval inclusions has dropped to $\leq N - 1$, a contradiction. Then we may assume that there is some $v \in V(G)$ so that, if $\phi(v) = [v_1, v_2]$, we have $x_2 < v_1 \leq z_2$.

Now, define a new function ϕ'' so that $\phi''(u) = \phi(u)$ for all $u \in V(G) \setminus \{x\}$, and $\phi''(x) = [z_1 - \epsilon', x_2]$, where $\epsilon' > 0$ has been chosen so that no interval $\phi(u)$, $u \in V(G) \setminus \{z\}$, has an endpoint in $[z_1 - \epsilon', z_1]$. Again, we obtain a contradiction unless there exists some $w \in V(G)$ so that, if $\phi(w) = [w_1, w_2]$, we have $z_1 \leq w_2 < x_1$. However, now z, x, w, and v form an induced claw in G. ☐

4. Prove that the complement of a bipartite graph is perfect.

Solution: (Virginia Johnson)

Proof. Let G be a bipartite graph. Since every induced subgraph of $\overline{G}$ is the complement of a subgraph in G, it is enough to show that $\chi(G) = \omega(G)$. Given any proper coloring of $\overline{G}$, notice that any color class has at most 2 vertices, since the largest clique in a bipartite graph is a K^2. The color classes of size two in this coloring form a matching in G. Notice also that given any matching M in G, there exists a subset U of the vertices in M of size $|M|$ that covers G (shown in the proof of König's Theorem, 2.1.1 in Diestel). The set $V(G) - U$ is independent, since for any two adjacent vertices one vertex must be in U. The set $V(G) - U$ is a clique in $\overline{G}$ of size $|G| - |M|$ so the matching corresponds to a coloring in $\overline{G}$ using $|G| - |M|$ colors. If M is a matching in G of maximum cardinality, then by König's Theorem, $|M|$ is equal to the minimum cardinality of a vertex cover U of its edges. Since $|U| = |G| - \alpha(G)$, M corresponds to a coloring in G with $|G| - |M|$ colors. But $|G| - |M| = |G| - |U| = |G| - (|G| - \alpha(G) = \alpha(G) = \omega(\overline{G})$. If $|M|$ is a maximum matching, then $|G| - |M|$corresponds to a coloring that uses the minimum number of color classes possible. Therefore, $\chi(\overline{G}) = \omega(\overline{G})$.

☐

5. Prove that the chromatic polynomial of a chordal graph has only real, integer roots. (Hint: First prove that every chordal graph has a *simplicial vertex*, i.e., $v \in V(G)$ so that $G[N(v)]$ is a clique.)

Solution: (Myself)

Suppose that every chordal graph has a simplicial vertex. Note that, if $G[N(v)] \simeq K^t$, then $p_G(\lambda) = (\lambda - t)p_{G-v}(\lambda)$, since we are free to color v with any color other than the t distinct colors used on $G[N(v)]$. Since p_G has only integer roots if G is complete, the conclusion follows from the hint by induction.

Claim 0.6. *Every chordal graph has a simplicial vertex.*

Proof. We proceed by induction on the number of "pastings" along complete subgraphs required to build up G from complete graphs, à la Diestel's Theorem 5.5.1. In fact, we show something stronger: every noncomplete chordal graph has two nonadjacent simplicial vertices. The base case is when G is itself complete, in which case the statement is trivial. Now, suppose G can be obtained by pasting together chordal graphs G_1 and G_2 along a set S so that $G_1[S] \simeq G_2[S] \simeq K^{|S|}$. By induction, G_1 possesses two nonadjacent simplicial vertices v_1 and v_2. In G, v_i is still simplicial if $v_i \notin S$. Since all elements of S are adjacent to one another, at least one of v_1 and v_2 is not an element of S, and therefore simplicial in G. Similarly, G_2 possesses a simplicial vertex w which is not an element of S and is therefore simplicial in G. Clearly, the two simplicial vertices found are nonadjacent, since neither is in S. ☐

6. Prove that, given two vertex colorings f and g of G with $D = \mathrm{col}(G) + 1$ colors, there exists a sequence of vertex D-colorings $f = c_1, c_2, \dots, c_{k-1}, c_k = g$ so that c_j differs from c_{j+1} at exactly one vertex for each $j = 1, \dots, k-1$.

Solution: (Andrew Dove)

Induction on the size of G. The $|G| = 0$ case is trivial. Assume that for graphs H such that $|H| < |G|$, there exists a sequence of vertex D-colorings $f = c_1, c_2, \dots, c_{k-1}, c_k = g$ so that c_j differs from c_{j+1} at exactly one vertex for each $j = 1, \dots, k-1$ for any two D-colorings f, g. Pick two D-colorings f, g. Pick a vertex v such that $\delta(v)$ is minimum so now because $d(v) < |D|$, there always exists a color in D that is not used by v or $N(v)$. For $G - v$ there exists a sequence of vertex D-colorings $f = c_1, c_2, \dots, c_{k-1}, c_k = g$ so that c_j differs from c_{j+1} at exactly one vertex for each $j = 1, \dots, k-1$. Set $c_i(v) = f(v)$ for all i. At every c_i such that there is an $x \in N(v)$ such that $c_i(x) = c_i(v)$ then between c_{i-1} and c_i, insert a coloring h_i such that h_i differs from c_{i-1} on v and $h_i(v)$ is the color that is not used by v or $N(v)$ in c_{i-1}. For all $j \geq i$ change $c_j(v)$ to be $h_i(v)$. In this way each coloring in this new sequence is a valid coloring of G and any two consecutive colorings in this new sequence differ at exactly one vertex of G.

7. Suppose G and H are nonempty and connected. Show that, if $G \Box H$ is planar iff either (a) one of G or H is K^1 and the other is planar, (b) one of G or H is K^2 and the other is outerplanar, or (c) one of G or H is P^r for some $r \geq 2$ and the other is either P^s for some $s \geq 2$ or a cycle.

Solution: (David Collins)

Proof. → a: If G is K^1, then $G \Box H$ is just H since G has no edges. Then since H is planar, $G \Box H$ must also be planar. b: Suppose G is outerplanar and H is K^2. Then $G \Box H$ is two copies of G with an edge between two vertices if they are the same vertex in G. Place two copies of G in the plane, with one a reflection of the other. Let a and a' be two copies of the same vertex. Then find a vertex b such that ab is an edge in the boundary of the outer face of G. Then we can connect b and b' as well, since they are both in the outer face. This cuts the outer face into 2 faces because we are now adding an edge to a connected graph. One of these faces is bounded by $abb'a$, and the other still has all of the other pairs. We can continue to add these edges, each time reducing the outer face, but leaving all the unpaired edges, until all the edges have been paired. Then we have an embedding of $G \Box H$ in the plane. c: If G and H are paths of length r and s respectively, then we can label the vertices of the paths starting at one end and going along. Then place the vertex (m, n) at the position (m, n) in the plane. (m, n) is connected to the vertices $(m-1, n), (m+1, n), (m, n-1)$, and $(m, n+1)$ as long as those are defined in the path. But we can easily draw lines between these points in the plane, we simply get an m by n grid. If H is a cycle and G is a path, then removing one edge from H turns it into a path. So $G \Box (H - e)$ is a grid, and we simply need to add the edges of $G \Box e$. These edges go from the left side of the grid to the right side. Starting with the bottom row of the grid, we can simply draw an edge going around the bottom of the grid. For each row above that, we can draw an edge going around the previously added ones, as they do not enclose either end of that row. This gives a planar embedding of $G \Box H$.

← First suppose that one of the graphs, wlog G, is K^1. Then $G \Box H$ is the same as H, so if $G \Box H$ is planar, then H must also have been planar.

Now suppose that one of the graphs, wlog G, is K^2. First we suppose that H contains a K^4 topological minor. Let a, b, c, and d be the four vertices. Since $G \Box H$ consists of 2 copies of H with edges between vertices that are the same, we can contract all the paths between a, b, c, and d in both copies of H, and find a $G \Box K^4$ as a minor in $G \Box H$. Then in this minor, we have that a, b, c, and d in the first copy of K^4 form a K^4. Clearly, aa' is an edge in this graph. Furthermore, we have the independent paths $a'b'b, a'c'c$, and $a'd'd$. So a, b, c, d, and a' for a K^5 topological minor in $G \Box H$, which would make $G \Box H$ not planar. Now suppose that H has a $K_{2,3}$ topological minor. Again we can contract edges, and find a $G\square K_{2,3}$ minor in $G\square H$. Call the vertices in the first copy of $K_{2,3}$, a_1, a_2, b_1, b_2, and b_3. Then these clearly form a $K_{2,3}$. If we include the vertex a_1', along with the independent paths $a_1'b_1'b_1, a_1'b_2'b_2$, and $a_1'b_3'b_3$, these six vertices form a $K_{3,3}$ minor in $G\square H$, which would make $G\square H$ non-planar. Thus if $G\square H$ is planar, then H cannot have a K^4 or a $K_{2,3}$ topological minor, and thus is outerplanar.

Now suppose that one of the graphs G is a path of length at least 2. Suppose H has a vertex of degree at least 3. Call this vertex a, and its neighbors b, c, and d. Then we have $P^2\square H$ as a subgraph of $G\square H$. Let H_1, H_2 and H_3 be the three copies of H in $P^2\square H$. Then consider the graph formed by the vertices a_1, a_2, a_3, b_2, c_2, and d_2. Clearly we have the edges a_2b_2, a_2c_2, and a_2d_2. We also have the independent paths $a_1b_1b_2, a_1c_1c_2, a_1d_1d_2, a_3b_3b_2, a_3c_3c_2$, and $a_3d_3d_2$. These six vertices with these paths creates a $K_{3,3}$ minor of $G\square H$, which would make $G\square H$ non-planar. So H cannot have a vertex of degree 3 or higher, so H is a path or a cycle.

Now suppose that neither G nor H is a path. Then neither are K^2, so they each contain P^2 as a subgraph. Thus if either graph has a vertex of degree 3 or higher, the above argument will work again. Therefore G and H are both cycles. As cycles, they have K^3 as a minor, so $G\square H$ has a $K^3\square K^3$ minor. This can be seen as three connected triangle. Call the vertices of a single triangle a, b, c, so that $K^3\square K^3$ is made up of a_i, b_i, c_i for $i = 1, 2, 3$. Then consider the vertices a_1, b_1, c_1, a_2, and a_3. Clearly we have the edges $a_1b_1, a_1c_1, a_1a_2, a_1a_3, a_2a_3$, and b_1c_1. By including the paths $a_2b_2b_1, a_2c_2c_1, a_3b_3b_1$, and $a_3c_3c_1$, we have a K^5 minor, which means that $G\square H$ is non-planar. $\square$

8. A graph G is called χ-unique if $p_G = p_H$ implies $G \simeq H$. Show that $K_{n,n}$ is χ-unique.

Solution: (Tatiana Orlova)

Proof. Consider $K_{n,n}$ – complete bipartite graph, and graph G, such that $p_G = p_{K_{n,n}}$. We want to show that $K_{n,n} \simeq G$. We will use the result known as the Whitney's Broken-cycle theorem, which states:

Let G be a graph on k vertices, having m edges, and let $\beta : E(G) \to 1, 2, \dots, m$ be any bijection. Then

$$p_G(\lambda) = \sum_{i=1}^{k} (-1)^{k-i} h_i(G) \lambda^i,$$

where $h_i(G)$ is the number of spanning subgraphs of G that have exactly $k - i$ edges and that contain no broken cycles with respect to β.

It follows from Whitney's results that:

1. $h_k = 1$ – the leading coefficient of a chromatic polynomial, which means that the degree of the polynomial equals the number of vertices in the vertex set of the graph. So, that in our case $p_G = p_{K_{n,n}}$ implies that $|V(G)| = |V(K_{n,n})| = 2n$;

2. $h_{n-1} = m$ is simply the number of edges in the graph. Thus, $p_G = p_{K_{n,n}}$ implies that $|E(G)| = |E(K_{n,n})| = n^2$.

Also, it is known that $\chi(G)$ is the smallest integer λ so that $p_G(\lambda) > 0$. Thus, $p_G = p_{K_{n,n}}$ also implies that $\chi(G) = \chi(K_{n,n})$.

Since graph G is such, that $|V(G)| = 2n$, $|E(G)| = n^2$, and $\chi(G) = \chi(K_{n,n})$ we can conclude that G must be bipartite. Denote by $V_1(G)$ and $V_2(G)$ partitions of its vertex set. Suppose that G is not isomorphic to $K_{n,n}$, meaning that $|V_1(G)| \neq |V_2(G)|$. WLOG suppose, that $|V_1(G)| > |V_2(G)|$, so that $|V_1(G)| = n + s$, where $0 < s \leq n - 1$. Then $|V_2(G)| = n - s$ and we have $n^2 = |E(G)| \leq (n + s)(n - s) = n^2 - s^2$, but we know that $n^2 > n^2 - s^2$. So, $|V_1(G)| = |V_2(G)| = n$ and therefore $K_{n,n} \simeq G$.♣ $\square$