Problem Set 5

MATH 776, Fall 2007, Cooper

Expiration: Tuesday December 11

Problems are ranked 0-5 based on their difficulty, indicated in parentheses next to the problem number. The number of points awarded for a **fully correct, rigorous** proof turned in before the expiration date above is $2^{\text{difficulty}}$. These problems are in addition to the problems in Diestel §4, which correspond to difficulties of 1,2, and 3, depending on whether they are marked with a $(-)$, no mark, or a $(+)$, respectively.

1. (3) Prove that the dual of every Hamiltonian planar graph is 4-colorable. (*Hamiltonian* means that there is a Hamiltonian cycle, a cycle containing every vertex exactly once.)

2. (2) Prove that every planar graph has a vertex of degree at most 5.

3. (3) Prove that, if a plane graph has all even degrees, then the faces can be 2-colored in such a way that faces with a common edge on their boundary receive different colors.

4. (3) Show that a connected plane graph G and its dual G^* have the same number of spanning trees.

5. (3) For which values of $r, s \geq 1$ is the complete bipartite graph $K_{r,s}$ planar? For which values of $r, s, t \geq 1$ is the complete tripartite graph $K_{r,s,t}$ planar? For which k is the hypercube Q_k planar?

6. (3) The "crossing number" $\text{CR}(G)$ of a graph G is the minimum number of crossings in any planar embedding of G. (Hence, $\text{CR}(G) = 0$ iff G is planar.) What are $\text{CR}(K_5)$, $\text{CR}(K_{3,3})$, and $\text{CR}(K_6)$?

7. (1) Show that the Petersen graph is not planar.

8. **(2)** How many isomorphism classes of nonplanar graphs on 6 vertices are there?

9. **(2)** Recall that a graph is "outerplanar" if it has a planar embedding with all of its vertices on a circle. Show that, if G is outerplanar, then $\chi(G) \leq 3$.

10. **(2)** Show that, if G is outerplanar with $n = |V(G)|$ and $m = |E(G)|$, then $m \leq 2n - 3$.

11. **(2)** A "fullerene" is a cubic planar graph all of whose faces are C_5 ("pentagons") or C_6 ("hexagons"). (They are named after the remarkable carbon allotropes, of which the "buckyball" – a soccerball-like molecule comprised of sixty carbons – is most famous.) In terms of $n = |V(G)|$, how many pentagons and hexagons goes a fullerene have? What is the least n for which a fullerene exists?