PROBLEM SET 4: DUE MONDAY 4/28/08

MATH 777, SPRING 2008, COOPER

Each problem is worth 20 points. Rigorous proofs are required for all claims, although elegance and concision are nearly as important. You may only use results we proved in class, or which can reasonably be considered prerequisite material for this class, unless otherwise stated. You will receive 5 bonus points for submitting your solutions in LaTeX. **NOTE:** The text has been updated on the website, so the problem numbers below refer to the *third* edition.

(1) Let $\mathcal{F}$ be a family of subsets of a finite set E. Call a subset $X \subseteq E$ a *transversal* of $\mathcal{F}$ if there is an injective function $f : X \to \mathcal{F}$ so that $x \in f(x)$ for each $x \in X$. Show that the set of all transversals forms an independence structure on E.
(2) Diestel §10, #2.
(3) Show that the n-cube Q_n is Hamiltonian (for $n \geq 2$).
(4) Prove Ore's Theorem: For $|G| = n \geq 3$, if $\deg(v) + \deg(w) \geq n$ for $v, w \in V(G)$ with $v \not\sim w$, then G is Hamiltonian. (Hint: Take an n-vertex counterexample maximal with respect to $\|G\|$. Add an edge, get a Hamiltonian cycle, subtract the edge, get a Hamiltonian path. How are the neighbors of the endpoints related?)
(5) Show that every graphic matroid is also a linear matroid.
(6) Prove that, if B_1 and B_2 are bases of a matroid M, and $e \in B_1 - B_2$, then there is an element f of $B_2 - B_1$ such that $(B_1 - e) \cup f$ and $(B_2 - f) \cup e$ are bases of M.
(7) Let M be a matroid with rank function r and closure operator $\overline{\cdot}$. Show that
   (a) $X \subseteq \overline{Y}$ and $\overline{Y} \subseteq \overline{X}$ implies $\overline{X} = \overline{Y}$.
   (b) $Y \subseteq \overline{X}$ implies $\overline{X \cup Y} = \overline{X}$.
   (c) $r(X \cup Y) = r(X \cup \overline{Y}) = r(\overline{X} \cup \overline{Y}) = r(\overline{X \cup Y})$.
   (d) $X \subseteq Y$ and $r(X) = r(Y)$ implies $\overline{X} = \overline{Y}$.