Problem Set 4

MATH 776, Fall 2007, Cooper

Expiration: Thursday November 15

Problems are ranked 0-5 based on their difficulty, indicated in parentheses next to the problem number. The number of points awarded for a **fully correct, rigorous** proof turned in before the expiration date above is $2^{\text{difficulty}}$. These problems are in addition to the problems in Diestel §3, which correspond to difficulties of 1,2, and 3, depending on whether they are marked with a $(-)$, no mark, or a $(+)$, respectively.

1. (0) Show that $c(G) + |E(G)| \geq |V(G)|$ for any graph G, where $c(G)$ is the number of components of G.

2. (1) Recall that $G \times H$ is defined to have vertex set $V(G) \times V(H)$ and edge set defined by $(u, v) \sim (u', v')$ iff $u \sim u'$ and $v \sim v'$. (This is known as the "weak direct product" or "tensor product" of G and H.) Prove that $G \times H$ is connected iff G is connected, H is connected, and at least of one G and H is nonbipartite.

3. (2) A connected k-regular bipartite graph is 2-connected.

4. (2) A directed graph (digraph) is called "strongly connected" if there is a directed path from any vertex to any other vertex. Prove that a tournament (an orientation of K_n) is strongly connected iff it contains a directed cycle that visits every vertex.

5. (2) Prove that a digraph is strongly connected iff there is at least one edge leaving each set $X \subset V(G)$ for $X \neq \emptyset$.

6. (2) A "cherry" is a pair of degree-one vertices in a graph with the same neighbor. Prove that, if G contains no cherries, then there are two adjacent vertices whose removal does not disconnect the graph. (Hint: Consider the longest path.)

7. (3) Prove that the distance function $d(\cdot,\cdot)$ and the length of the *longest* path connecting two vertices, denoted $D(\cdot,\cdot)$, are metrics.

8. (2) Prove that, in any connected graph G, any two paths of maximum length have a point in common.

9. (3) Prove that, if T_1 and T_2 are two spanning trees of G, then T_2 can be obtained from T_1 by a sequence of "swaps", always preserving the property of being a spanning tree. A swap consists of removing one edge from the tree and replacing it with another.

10. (4) Call two edges of a 2-edge-connected graph G "equivalent" if they are equal or their removal disconnects the graph. Show that:

a.) Equivalence is an equivalence relation.

b.) All edges of an equivalence class lie on a cycle.

c.) Removing the edges of an equivalence class P, the components of the remaining graph are 2-edge connected.

d.) Contracting the components of $G - P$ yields a cycle.

11. (3) Let A and B be disjoint subsets of $V(G)$ and assume that any set X which has a point from each (A, B)-path has at least k elements. Show that there are k vertex-disjoint (A, B)-paths.

12. (3) Show that $\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$, where $G \Box H$ is the Cartesian product, defined to have vertex set $V(G) \times V(H)$ and edge set $\{\{(a, b), (c, d)\} : ((a = c) \wedge (b \sim_H d)) \vee ((a \sim_G c) \wedge (b = d))\}$.