Problem Set 3

MATH 776, Fall 2007, Cooper

Expiration: Thursday October 25

Problems are ranked 0-5 based on their difficulty, indicated in parentheses next to the problem number. The number of points awarded for a **fully correct, rigorous** proof turned in before the expiration date above is $2^{\text{difficulty}}$. These problems are in addition to the problems in Diestel §2, which correspond to difficulties of 1,2, and 3, depending on whether they are marked with a $(-)$, no mark, or a $(+)$, respectively. Note that *you may use the the results of previous problems even if you cannot prove them.*

1. ($4.3923\ldots$**)** A matrix is called *doubly stochastic* if its row and columns each add up to 1, and its entries lie in the interval $[0,1]$. A *permutation matrix* is a doubly stochastic matrix all of whose entries are 0 or 1. Recall the notation $[n] = \{1, \ldots, n\}$ and the "join" operation

$$G * H = (V(G) \,\dot\cup\, V(H), E(G) \,\dot\cup\, E(H) \,\dot\cup\, (V(G) \times V(H)))\,.$$

a. (1) Show that a doubly stochastic matrix is square; that muliplying a permutation matrix by a vector permutes the coordinates of that vector; and that each permutation of the coordinates arises from some permutation matrix in this way. In other words, if we denote the matrix representing a permutation by M_π, then $\pi \mapsto M_\pi$ is a bijection between the set of permutations and the set of permutation matrices.

b. (2) Suppose we assign a weight $\mathrm{wt}(i,j) \in [0,1]$ to each edge of $G = K_{n,n} = [n] * [n]$ so that the sum of all the weights at each vertex is 1. Show that G contains a 1-factor all of whose edges have positive weight. (Hint: Imitate the proof that regular bipartite graphs have a 1-factor.)

c. (2) Show that, for each nonzero entry (i, j) of a doubly stochastic matrix $A \in \mathbb{R}^{n\times n}$, there is some permutation $\pi = \pi^A_{i,j}$ with $\pi(i) = j$ so that the set of $(k, \pi(k))$ entries in A for $k \in [n]$ are positive. (Hint: A permutation can be thought of as a matching in $K_{n,n}$ weighted by the entries of A, where each block in the bipartition are thought of as the numbers $[n]$, one block of the bipartition corresponding to the indices of the rows of A and the other the columns.)

d. (0) Show that $A - xM_{\pi^A_{i,j}}$ has nonnegative entries, where x is the minimum positive entry of A and $A(i, j) > 0$.

e. (1) Show that, if A is doubly stochastic with minimum positive entry $A(i, j) = x$ where $0 < x < 1$, then the matrix $A' = (A - xM_{\pi^A_{i,j}})/(1 - x)$ is doubly stochastic.

f. (2) Apply induction on the number of nonzero entries to prove that every $n \times n$ doubly stochastic matrix A can be written as

$$A = \sum_{\pi\in S_n} \lambda_\pi M_\pi$$

for some $(\lambda_\pi)_\pi \in [0, 1]^{n!}$ with $\sum_\pi \lambda_\pi = 1$.

g. (1) Show that the set of doubly stochastic matrices is a polytope P whose vertices are the permutation matrices (think of $\mathbb{R}^{n\times n}$ as an n^2-dimensional vector space). (A polytope is the convex hull of a set of vertices $\{v_i\}^t_{i=1}$, i.e., the set of points of the form $\sum^t_{i=1} \lambda_i v_i$ where $(\lambda_i)^t_{i=1} \in [0, 1]^t$ and $\sum^t_{i=1} \lambda_i = 1$.) This is the Birkhoff-von Neumann Theorem.

h. (1) Show that P is $(n - 1)^2$-dimensional, i.e., there is a hyperplane of dimension $(n-1)^2$ in the n^2-dimensional space $\mathbb{R}^{n\times n}$ that contains P, and no lower-dimensional hyperplane has this property. P is known as the "Birkhoff Polytope."

2. (2) Suppose G is bipartite and connected. Let $\rho(G)$ denote the edge-covering number of G, the minimum number of edges in G so that every vertex is incident to at least one of them; and let $\alpha(G)$ denote the independence number of G, the maximum number of independent vertices in G. Prove that $\alpha(G) = \rho(G)$.

3. (3) Show that the following are equivalent for a bipartite graph G with bipartition $\{A, B\}$:

(a.) G is connected and every edge of G is contained in a 1-factor.

(b.) For any $x \in A$ and $y \in B$, $G - x - y$ has a 1-factor.

(c.) $|A| = |B|$ and, for each $\emptyset \neq X \subsetneq A$, $|N(X)| > |X|$.

Such a graph is known as "elementary bipartite."

4. (4) Show that a bipartite graph G is elementary (see previous problem) iff it can be written

$$G = G_0 \cup P_1 \cup \cdots \cup P_k$$

where G_0 consists of two points and the edge between them, and P_i is an odd path whose (vertex) intersection with $G_0 \cup P_1 \cup \cdots \cup P_{i-1}$ consists of exactly its two endpoints.

5. (2) Show that an edge-minimal elementary bipartite graph G has a vertex of degree 2.

6. (2) A bipartite graph G is the union of $\Delta(G)$ matchings. (Hint: Add vertices and edges until the graph is regular.)

7. (3) A "latin rectangle" of order n is a matrix with entries in $[n]$ so that each the entries in any row or column are distinct. Define $N(i)$ to be the number of times the integer i appears in such a matrix. Show that an $r \times s$ latin rectangle can be extended to an $n \times n$ latin square of order n iff $N(i) \geq r + s - n$ for each i. (Hint: Add rows/columns one at a time, i.e., induction. Consider the bipartite graph whose color classes are $[n]$ and the sets S_i, $1 \leq i \leq r$, consisting of those elements of $[n]$ not appearing in row i; and an edge from j to S_i if $j \in S_i$.)

8. (2) A graph is "independence-regular" if every maximal independent set of vertices has the same cardinality. Prove that a tree is independence-regular iff every vertex is either a leaf or has exactly one leaf neighbor.