Problem Set 2

MATH 776, Fall 2007, Cooper

Expiration: Thursday September 27

Problems are ranked 0-5 based on their difficulty. The number of points awarded for a **fully correct, rigorous** answer turned in before the expiration date above is $2^{\text{difficulty}}$. These problems are in addition to the problems in Diestel §1, which correspond to difficulties of 1,2, and 3, depending on whether they are marked with a $(-)$, no mark, or a $(+)$, respectively.

1. (3) Which graphs are isomorphic to their line graphs?

2. (2) What are the diameter, girth, and radius of K_n? C_n? P_n? Q_n? The n-star S_n (i.e., $S_n = E_1 * E_n = K_{1,n}$, where E_j is the empty graph on j vertices)?

3. (3) Prove that every graph has two vertices of the same degree. (To upgrade this problem to a difficulty of 4, also determine which graphs have exactly one pair of vertices of the same degree.)

4. (1) Prove that, if $v, w \in V(G)$ are the only vertices of G of odd degree, there is a v-w Eulerian walk in G (i.e., a walk starting at v and ending at w that visits every edge exactly once).

5. (2) Prove that Prüfer codes provide a bijection from $[n]^{n-2}$ to the set of all labeled trees on n vertices.

6. (2) Prove the following generalization of Cayley's formula: The number of labeled trees on n vertices with degree sequence $d_1, \ldots, d_n$ is

$$\frac{(n-2)!}{\prod_{j=1}^{n}(d_j - 1)!}.$$

7. (1) What does the Depth-First-Search (DFS) tree (see problem 1.17 in Diestel) look like in K_n?

8. (2) Show that the central vertices in a tree are either a single vertex, or two adjacent vertices.

9. (1) Let $T_1, \ldots, T_r$ be trees on disjoint sets of vertices and $V = \bigcup_j V_j$. How many trees are there on V which contain all of the T_j?

10. (3) Let W_n denote the number of isomorphism classes of trees on n points. Prove that, for $n \geq 50$, $W_n > 2^n$. (To upgrade this to a difficulty of 4, also show that $W_n < 4^n$.)

11. (4) Let the "ladder graph" L_n be defined as follows: $L_n = P_1 \cup P_2 \cup \mathcal{E}$, where P_1 and P_2 are disjoint paths of length $n-1$, and $\mathcal{E}$ is the set of n edges connecting vertex j of the first path to vertex j of the second path for each $j \in [n]$. How many perfect matchings (i.e., maximal 1-regular subgraphs) does L_n have?