

# Final Exam

MATH 777, Spring 2008, Cooper

Thursday May 1

The number of points awarded for a **fully correct, rigorous** proof is 12; less for errors, omissions, or lack of clarity. It is very important that your writing be legible and your arguments concise and elegant – so think before you commit anything to paper. You may use any theorem covered in class or on a problem set, and any result that one could reasonably expect to know from an undergraduate mathematics education. (If you are unsure about something, ask.)

1. Let  $M$  be the graphical matroid corresponding to the cycle graph  $C_n$ . Show that the Whitney numbers of  $\mathcal{L}(M)$  are log-concave.
2. What is the chromatic polynomial  $P_G(\lambda)$  of  $K_{2,s}$ , the complete bipartite graph with block sizes 2 and  $s$ ? (Give a closed form.)
3. Given a poset  $P = (S, \preceq)$ , the *comparability graph*  $G$  of  $P$  is the simple graph with vertex set  $S$  and  $x \sim y$  whenever  $x$  and  $y$  are comparable (i.e.,  $x \preceq y$  or  $y \prec x$ ). Show that  $G$  is perfect.
4. What is the flow number  $\varphi(G)$  of the complete balanced bipartite graph  $K_{s,s}$ ?
5. Show that, for any  $c > 0$ , there exists an  $a > 0$  so that, if  $G$  is any graph on  $n$  vertices with at most  $an^3$  triangles, then it is possible to remove  $cn^2$  edges from  $G$  to make it triangle free.
6. Suppose that  $G$  is tripartite, with blocks  $A_i$ ,  $i \in [3]$ , of size  $n$ , and that  $(A_i, A_j)$  is  $\epsilon$ -regular of density  $1/2$  for  $i \neq j$ . Show that  $G$  contains at least  $cn^3$  triangles for some  $c > 0$ .
7. Let  $S \subset \Gamma$  for some finite commutative group  $\Gamma$  in which every element has order 2 (i.e.,  $g \in \Gamma \Rightarrow g + g = 0$ ). Show that, if  $S$  is sum-free (i.e.,  $a, b \in S$ ,  $a \neq b$ , implies  $a + b \notin S$ ) then  $|S| \leq |\Gamma|/2 + 1$ . Hint: Consider the graph  $G$  with vertex set  $\Gamma$  and  $x \sim y$  whenever  $x + y \in S$ .
8. Show that, if a graph  $G$  on  $n$  vertices satisfies

$$\|G\| \geq \frac{n^2 - 3n + 6}{2} \geq 3,$$

then  $G$  is Hamiltonian.

9. Show that a tournament (a digraph  $(V, E)$  so that  $(x, y) \in E$  iff  $(y, x) \notin E$  for  $x \neq y$ ) is Hamiltonian (i.e., contains a directed spanning cycle) iff it is strongly connected (i.e., has a directed path from any vertex to any other).
10. Recall that the homogeneous number  $\text{hom}(G) = \max\{\alpha(G), \omega(G)\}$ . Show that, if  $G$  is perfect, then  $\text{hom}(G) \geq \sqrt{|G|}$ .
11. Show that, for any element  $e$  of a matroid  $M$ , the following are equivalent:
1.  $e$  is in every basis.
  2.  $e$  is in no circuits.
  3. If  $X \subset E(M)$  and  $e \in \overline{X}$ , then  $e \in X$ .
  4.  $r(E(M) - e) = r(E(M)) - 1$
  5.  $\{e\}$  is closed.
  6. If  $I$  is an independent set, then so is  $I \cup e$ .
12. Show that a lattice  $\mathcal{L}$  is geometric if, for all  $x, y \in \mathcal{L}$ , the following holds: “If  $x$  and  $y$  cover  $x \wedge y$ , then  $x \vee y$  covers both  $x$  and  $y$ ”.
13. Using Problem 8 above, compute  $\text{ex}(n, P_{n+1})$ .
14. Fix a graph  $G$  and  $\epsilon > 0$ . Show that there are only finitely many  $k$  so that  $G$  has more than  $(1 + \epsilon)$  times as many  $(k + 1)$ -flows as  $k$ -flows. (Be careful about the details here!)
15. Recall that  $U(n, k)$ , the uniform matroid, has basis system  $\mathcal{B} = \binom{[n]}{k}$ . Show that  $U(4, 2)$  is not graphic.
16. What is  $R(K_3, K_4 - e)$ ?
17. Suppose that  $G$  is cubic and Hamiltonian. Show that  $E(G)$  can be partitioned into three disjoint 1-factors.
18. Show that, if  $G$  is connected and every edge of  $G$  is contained in a triangle, then  $L(G)$  (the line graph of  $G$ ) is Hamiltonian.