Homework 4: Due Thursday July 1

Math 574, Prof. Cooper, Summer I 2010

Each problem is worth 10 points, for a total of 100 points. Any points earned on them can only help your homework grade in the following sense: if your first three homework grades were a, b, and c, and you score a d on this homework, then your final homework grade will be

$$100(a + b + c + d)/(300 + d).$$

This is equivalent to treating this homework as a 100%, but giving it a weight of $100d/(300 + d)\%$ instead of just 25% (whereas each of the previous homeworks are equal in weight).

Note that the problem statements are cumulative. Nonetheless, you may use the statement of problem n to solve problem m whenever $n < m$, regardless of whether you solved the former. As usual, you are expected to work alone on this problem set. *Hint*: Many of the problems can be solved using the Principle of Mathematical Induction.

Define the *Fibonacci sequence* F_0, F_1, F_2, ... recursively by setting $F(0) = 0$, $F(1) = 1$, and $F(n + 2) = F(n + 1) + F(n)$ for $n \geq 0$.

1. Compute the first 10 terms of the sequence.
2. Show (using induction) that $F_n = [\phi^n - (-1/\phi)^n]/\sqrt{5}$, where $\phi = (1 + \sqrt{5})/2$ is the so-called "golden ratio".
3. Show that $\lim_{n\to\infty} F_{n+1}/F_n = \phi$.
4. For which n is F_n even? State a conjecture and prove it.
5. Prove that $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.
6. Prove that $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$.
7. Show that the number of ways to tile an $n \times 2$ rectangle with dominoes (2×1 rectangles) is F_{n+1} for $n \geq 1$.
8. Let $s(x)$ be the power series given by $\sum_{n=0}^{\infty} F_n x^n$. (This is called the "generating function" of the sequence $\{F_n\}_{n=0}^{\infty}$.) Show that $s(x)(1 - x - x^2) = x$, so that $s(x) = x/(1 - x - x^2)$.
9. What is the radius of convergence of the series $s(x)$ given above?
10. Use the formula for $s(x)$ from Problem 8 to provide another proof of the formula $F_n = [\phi^n - (-1/\phi)^n]/\sqrt{5}$ from Problem 2. (Hint: Compute the partial fraction decomposition and write $s(x)$ as the sum of two infinite geometric series.)