Math 574 Final Exam Solutions

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1. Express each sentence in the predicate calculus using only the predicates and domain given. (You may also use any algebraic expression involving the variables, and the name of any particular element of the domain.)

   (a) "For every rational number x, there is an integer y so that xy is an integer." Domain $= \mathbb{Q}$, $I(x) =$ "x is an integer."

   (b) "For any two distinct integers x and y, $|x-y| \leq 1$ if and only if there exist no integers strictly between x and y." Domain $= \mathbb{Z}$.

   (c) "Meals served on Friday taste better than those served on any other day." Domain of $x, y =$ meals, domain of $d =$ days of the week, $S(x,d) =$"x is served on day d", $B(x,y) =$"x tastes better than y".

   (d) "If a user does not enter his/her password correctly, anyone with the same username is denied access." Domain = users, $P(x) =$"x enters his/her password correctly", $S(x,y) =$"x and y have the same username", $D(x) =$"x is denied access."

   (a) $\forall x \exists y (I(y) \wedge I(xy))$

   (b) $\forall x \forall y\, ((|x-y| \leq 1) \leftrightarrow (\neg \exists z[((x < z) \wedge (z < y)) \vee ((y < z) \wedge (z < x))]))$

   (c) $\forall x\, [(S(x, \text{Friday}) \rightarrow \forall y((\exists d (d \neq \text{Friday}) \wedge S(y,d)) \rightarrow B(x,y))]$

   (d) $\forall x\, [\neg P(x) \rightarrow \forall y (S(x,y) \rightarrow D(y))]$

2. Which of the following statements are true? In each case, indicate the reason for your answer.

   (a) $\{\emptyset\} \in P(\{\{\emptyset\}\})$ (Recall that $P(S)$, for a set S, is the "power set" of S, the set of all subsets of S.)

   (b) For any sets A, B and C, $(A-B) \cup (B-C) = A \cup B - A \cap B \cap C$.

   (c) The set of one-to-one functions from a finite set A to a finite set B is nonempty if and only if $|A| \leq |B|$.

   (d) If $R \subseteq A \times A$ and, for every $a \in A$, there is a $b \in A$ so that $(a,b) \in R$, then for every $b \in A$, there is an $a \in A$ so that $(a,b) \in R$.

   (a) **False**. $S \in P(T)$ iff $S \subseteq T$. However, $\emptyset \notin \{\{\emptyset\}\}$, so $\{\emptyset\} \not\subseteq \{\{\emptyset\}\}$ and $\{\emptyset\} \notin P(\{\{\emptyset\}\})$.

   (b) **False**. Suppose that there exists an $x \in \overline{A} \cap B \cap C$. Then $x \notin A \rightarrow x \notin A - B$ and $x \in C \rightarrow x \notin B - C$, so $x \notin (A-B) \cup (B-C)$. However, $x \in B \rightarrow x \in A \cup B$, but $x \notin A \rightarrow x \notin A \cap B \cap C$, so $x \in A \cup B - A \cap B \cap C$.

(c) **True**. Suppose $|A| \leq |B|$. Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$, where all the a_j's are distinct and all the b_j's are distinct. Then we may define a function $f : A \to B$ by defining $f(a_j) = b_j$ for each j with $1 \leq j \leq m$; this expression makes sense because $|A| = m \leq n = |B|$, so $1 \leq j \leq m$ implies $1 \leq j \leq n$. Furthermore, if $f(a_i) = f(a_j)$, then $b_i = b_j$, which implies $i = j$. On the other hand, if $|A| > |B|$, suppose there exists a one-to-one function $f : A \to B$. Then $|f(A)| = |A|$, since f provides a bijection between A and $f(A)$. However, $f(A) \subset B$, so $|A| = |f(A)| \leq |B|$, contradicting our assumption that $|A| > |B|$. Therefore, there exists no such f.

(d) **False**. Consider $A = \{1, 2\}$, $R = \{(1,1), (2,1)\}$. Then, for all $a \in A$, there exists a $b \in A$ so that $(a, b) \in R$ – namely, $b = 1$. However, for $b = 2$, there is no $a \in A$ so that $(a, b) \in R$.

3. Let P_n denote the n-th Padovan number, i.e., $P(0) = 1$, $P(1) = 1$, $P(2) = 1$, and $P(n) = P(n-2) + P(n-3)$ for $n \geq 3$. Prove that

$$\sum_{m=0}^{n} P(m) = P(n+5) - 2.$$

(Hint: First prove that $\sum_{m=0}^{n} P(m) = P(n+2) + P(n+3) - 2$.)

Proof. We proceed by (weak) induction on n to prove the claim in the hint first. Note that $P(3) = 2$, $P(4) = 2$, and $P(5) = 3$.

Base Case $n = 0$: $\sum_{m=0}^{0} P(m) = P(0) = 1$ and $P(2) + P(3) - 2 = 1 + 2 - 2 = 1$.

Base Case $n = 1$: $\sum_{m=0}^{1} P(m) = P(0) + P(1) = 2$ and $P(3) + P(4) - 2 = 2 + 2 - 2 = 2$.

Base Case $n = 2$: $\sum_{m=0}^{2} P(m) = P(0) + P(1) + P(2) = 3$ and $P(4) + P(5) - 2 = 2 + 3 - 2 = 3$.

Induction Step: Suppose that $\sum_{m=0}^{n} P(m) = P(n+2) + P(n+3) - 2$ for some $n \geq 3$. Then

$$\begin{aligned} \sum_{m=0}^{n+1} P(m) &= \sum_{m=0}^{n} P(m) + P(n+1) \\ &= P(n+2) + P(n+3) - 2 + P(n+1) \\ &\qquad \text{by the Inductive Hypothesis} \\ &= (P(n+1) + P(n+2)) + P(n+3) - 2 \\ &= P(n+4) + P(n+3) - 2 \\ &\qquad \text{by the definition of } P(k). \end{aligned}$$

Now that we've established the claim in the hint, we prove the original statement:

$$\sum_{m=0}^{n} P(m) = P(n+3) + P(n+2) - 2$$
$$= P(n+5) - 2,$$

again, by the definition of the Padovan sequence. □

4. How many integers k with $1 \leq k \leq 999$

   (a) do not contain the digit 1 in base 10?

   (b) are divisible by 5 but not by 6?

   (c) are such that any set with k elements has more than 27000 subsets consisting of exactly three elements?

   (d) contain two consecutive 1's or two consecutive 0's in base 2?

You do not have to provide the reasons for your answers for this problem.

   (a) Let S be the set of integers k with $1 \leq k < 1000$ not containing the digit 1 in base 10. We partition S into three sets: $A = \{k : 1 \leq k < 10\}$, $B = \{k : 10 \leq k < 100\}$, and $C = \{k : 100 \leq k < 1000\}$. In A, each k has a single digit, which can be anything but 0 or 1. Therefore, $|A| = 8$. In B, each k has two digits, the first of which cannot be 0 or 1, and the second of which cannot be 1. Therefore $|B| = 8 \cdot 9 = 72$. In C, each k has three digits, the first of which cannot 0 or 1, and the second and third of which cannot be 1. Therefore, $|C| = 8 \cdot 9 \cdot 9 = 648$. Since S is the disjoint union of A, B, and C, $|S| = |A| + |B| + |C| = 728$.

   (b) Let A denote the set of those k with $1 \leq k < 1000$ which are divisible by 5, and let B denote the set of those k with $1 \leq k < 1000$ which are divisible by 6. Then $|A| = \lfloor 999/5 \rfloor = 199$ and

$$|A \cap B| = |\{k : 1 \leq k \leq 1000 \wedge 30|k\}| = \left\lfloor \frac{999}{30} \right\rfloor = 33.$$

   Therefore, the desired set, $A - B$, has cardinality $|A| - |A \cap B| = 199 - 33 = 166$.

   (c) The number of subsets of a set with k elements containing exactly three elements is $C(n, 3)$. Since $C(n, 3) = n(n-1)(n-2)/6$, we wish to know for which n does the following hold:

$$\frac{n(n-1)(n-2)}{6} > 27000.$$

   Multiplying both sides by 6 yields $n(n-1)(n-2) > 162000$. Since $55 \cdot 54 \cdot 53 = 157410$ and $56 \cdot 55 \cdot 54 = 166320$, there are precisely $999 - 55 = 944$ integers with the desired property.

(d) If a positive integer, when written in binary, has no consecutive 0's or 1's, it must simply alternate 0's and 1's. It must also start with a 1. Therefore, it belongs to the sequence 1_2, 10_2, 101_2, 1010_2, 10101_2, 101010_2, 1010101_2, 10101010_2, 101010101_2, 1010101010_2, 10101010101_2, 101010101010_2, ... However, $1010101010_2 = 512_{10} + 128_{10} + 32_{10} + 8_{10} + 2_{10} = 682_{10}$, whereas $10101010101_2 > 1024_{10} > 999_{10}$. Therefore, there are 10 such k. The problem asks for those integers which *do* have two consecutive 0's or 1's in base 2, so there are 999-10 = 989 of them.

5. For which elements of $\mathbb{N}$ is it true that $3n + 5 < 2^n$? Prove your answer.

Proposition 1. *For $n \in \mathbb{N}$, $3n + 5 < 2^n$ if and only if $n \geq 5$.*

Proof. First, we verify that $3n+5 \geq 2^n$ for $n \in \{0, 1, 2, 3, 4\}$: $3\cdot0+5 = 5 \geq 1 = 2^0$, $3\cdot1+5 = 8 \geq 2 = 2^1$, $3\cdot2+5 = 11 \geq 4 = 2^2$,$3\cdot3+5 = 14 \geq 8 = 2^3$, and $3 \cdot 4 + 5 = 17 \geq 16 = 2^4$. Now, we show that $3n + 5 < 2^n$ for $n \geq 5$ by induction. For the base case, we verify that $3 \cdot 5 + 5 = 20 < 32 = 2^5$. Then, for the induction step, suppose that $3n + 5 < 2^n$. Then

$$\begin{aligned} 3(n+1) + 5 &= 3n + 5 + 3 \\ &< 2^n + 3 \\ &\qquad \text{by the Inductive Hypothesis} \\ &< 2^n + 2^n \\ &\qquad \text{because } n \geq 5 \to 2^n \geq 32 > 3 \\ &= 2^{n+1}. \end{aligned}$$

Since this is the desired conclusion for $n + 1$, the statement follows. □

6. Prove that the number of 4-permutations of a set with n elements is divisible by 4 for all $n \in \mathbb{N}$.

Proof. Note that the number of 4-permutations of a set with n elements is given by

$$P(n, 4) = \frac{n!}{(n-4)!} = n(n-1)(n-2)(n-3).$$

We offer five different proofs that this quantity is always divisible by 4. First, we proceed by induction. The base case $n = 0$ clearly holds, since $P(0, 4) = 0$. Now, for the inductive step, suppose that $n(n-1)(n-2)(n-3) = 4k$ for some integer k. Then

$$\begin{aligned} P(n+1, 4) &= (n+1)n(n-1)(n-2) \\ &= n(n-1)(n-2)(n-3+4) \\ &= n(n-1)(n-2)(n-3) + 4n(n-1)(n-2) \\ &= 4k + 4l, \end{aligned}$$

where $l = n(n-1)(n-2)$ is clearly an integer. Therefore, $P(n+1,4)$ is divisible by 4.

For the second proof of this fact, note that, for any four consecutive integers n, $n-1$, $n-2$, and $n-3$, two are even and two are odd. Therefore, the product of these four quantities is divisible by $2 \cdot 2 = 4$.

For the third proof of this fact, note that, for any four consecutive integers n, $n-1$, $n-2$, and $n-3$, exactly one must be divisible by 4. To see this, simply consider writing $n = 4k + r$, where $0 \le r < 4$, i.e., r is the remainder when n is divided by 4. Then $n - r = 4k$. Since the four quantities n, $n-1$, $n-2$, and $n-3$ includes $4k$, their product is divisible by 4.

For the fourth proof of this fact, note that we may group the set of 4-permutations into "cyclically equivalent" classes: abcd, bcda, cdab, and dabc are grouped together, for example. Since all four elements are distinct, each such group has exactly four elements. Therefore, if k is the total number of groups, $P(n,4) = 4k$, i.e., $P(n,4)$ is divisible by 4. (Perhaps this is unconvincing when $n < 4$; it is easy to check, however, that $P(0,4) = P(1,4) = P(2,4) = P(3,4) = 0$ are all divisible by 4.)

For the fifth proof of this fact, note that $C(n,4) = P(n,4)/4! = P(n,4)/24$. Since $C(n,4)$ is an integer,

$$P(n,4) = 24C(n,4) = 4(6C(n,4)),$$

and we may conclude that 4 divides $P(n,4)$. □

7. Prove that the following statements about an integer n are equivalent. Recall that the "parity" of an integer n is the answer to the question, "Is n even or odd?". You may assume that every integer is either even or odd, and no integer is both.

   (a) n is odd.
   (b) n^3 is odd.
   (c) $(-1)^n = -1$.
   (d) For all integers m, $m+n$ and m have opposite parities.
   (e) For all integers m, mn and m have the same parity.

   Proof.

   $(a) \to (b)$**:** If n is odd, it can be written as $n = 2k+1$ for some integer k. Then

   $$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 = 2l + 1,$$

where $l = 4k^3 + 6k^2 + 3k$ is clearly an integer. Therefore, n^3 is odd as well.

$(b) \to (a)$: We prove the contrapositive. If n not odd, then it is even, so it can be written as $n = 2k$ for some integer k. Then

$$n^3 = (2k)^3 = 8k^3 = 2(4k^3) = 2l,$$

where $l = 4k^3$ is clearly an integer. Therefore, n^3 is even (i.e., not odd) as well.

$(a) \to (c)$: If n is odd, it can be written as $n = 2k + 1$ for some integer k. Then

$$(-1)^n = (-1)^{2k+1} = [(-1)^2]^k \cdot (-1) = 1^k \cdot (-1) = 1 \cdot (-1) = -1.$$

$(c) \to (a)$: We prove the contrapositive. If n is not odd, then it is even and can be written as $n = 2k$ for some integer k. Then

$$(-1)^n = (-1)^{2k} = [(-1)^2]^k = 1^k = 1 \neq -1.$$

$(a) \to (d)$: Suppose n is odd, so it can be written as $n = 2k + 1$ for some integer k. Then, either m is even or m is odd. If m is even, then $m = 2l$ for some integer l, so $m+n = 2l+2k+1 = 2(l+k)+1$, which is odd, unlike m. If m is odd, then $m = 2l + 1$ for some integer l, so $m + n = 2k + 1 + 2l + 1 = 2(k + l + 1) = 2p + 1$, where $p = k + l + 1$ is clearly an integer. Therefore, $m + n$ is even, unlike m. Therefore m and n have opposite parity.

$(d) \to (a)$: We prove the contrapositive. Suppose n is not odd, so it is even and can be written as $n = 2k$ for some integer k. Then taking $m = 0$ yields $m + n = 2k$, which is even, i.e., the same parity as m. Therefore, there exist m so that $m + n$ and m do not have opposite parities.

$(a) \to (e)$: Suppose n is odd, so it can be written as $n = 2k + 1$ for some integer k. Then, either m is even or m is odd. If m is even, then $m = 2l$ for some integer l, so $mn = 2l(2k + 1) = 2(2lk + l)$, which is even (like m itself). If m is odd, then $m = 2l+1$ for some integer l, so $mn = (2k+1)(2l+1) = 4kl+2k+2l+1 = 2(2kl+k+l)+1 = 2p+1$, where $p = 2kl+k+l$ is clearly an integer. Therefore, mn is odd, like m itself. Therefore m and n have the same parity.

$(e) \to (a)$: We prove the contrapositive. Suppose n is not odd, so it is even and can be written as $n = 2k$ for some integer k. Then taking $m = 1$ yields $mn = 2k$, which is even, i.e., the opposite parity of m. Therefore, there exist m so that mn and m do not have the same parity.

□

8. Suppose that a valid US phone number consists of ten digits satisfying the following restrictions: the first three digits must be drawn from a pool of 269 three-digit area codes; the fourth digit cannot be a 0 or a 1; the fifth and sixth digit cannot both be 1's or both be 0's; and no digit can occur three or more times among the last four digits. How many valid US phone numbers are there? (You do not have to compute any product involving two numbers both of which have two or more digits in base 10.)
Solution: There are 269 ways to choose the first three digits, 8 ways to choose the fourth digit, and $100 - 2 = 98$ ways to choose the fifth and sixth digits. Finally, there are 1000 possible combinations for the last four digits, but some of them repeat a digit more than twice. If a digit appears exactly three times, there are $C(4, 3) = 4$ choices of the three places it could be, 10 choices for which digit it is, and 9 choices for the remaining digit. There are exactly 10 ways for a digit to appear four times: 0000, 1111, ..., 9999. Therefore, the total number of valid phone numbers is:

$$269 \cdot 8 \cdot 98 \cdot (1000 - 4 \cdot 10 \cdot 9 - 10) = 210896 \cdot 630 = 132864480.$$

(Note that this is only about a third of the U.S. population!)

9. For each of the following statements, determine its truth value for each domain. Recall that $\mathbb{Q}^+$ denotes the positive rational numbers. You do not have to provide reasons for your answers to this problem.

	$\mathbb{Z}$	$\mathbb{R}$	$\mathbb{Q}^+$
$\forall x \forall y((xy = 0) \to ((x = 0) \vee (y = 0)))$	**T**	**T**	**T**
$\forall x \exists y(x^2 > y^2)$	**F**	**F**	**T**
$\forall x((x \neq 0) \to \exists y \forall z(xyz = z))$	**F**	**T**	**T**
$\exists x \neg \exists y(2xy = x + y)$	**T**	**T**	**T**

From left-to-right and top-to-bottom:

1 & 2. If $xy = 0$, then $x = 0$ or $y = 0$. (This is just a basic property of multiplication.)

3. Here, $0 \notin \mathbb{Q}^+$, so the antecedent is always false and the statement is vacuously true.

4 & 5. If $x = 0$, then $x^2 = 0$, and, for any y, $y^2 \geq 0$. Therefore, there is no y so that $y^2 < x^2$.

6. We can take $y = x/2$. If $x = p/q$ is a positive rational, then $y = p/(2q)$ is as well. Furthermore,

$$y^2 = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4} < x^2,$$

since $x^2 > 0$ for all $x \in \mathbb{Q}^+$.

7. Consider $x = 2$. If $xyz = z$ for all z, then it holds for $z = 1$ in particular, and so $xy = 1$. However, $2y = 1$ does not have a solution in $\mathbb{Z}$.

8 & 9. Let $y = 1/x$; this expression makes sense because $x \neq 0$. Then $xyz = (x/x)z = z$ for any z in the domain.

10. Consider $x = 2$. Then $2xy = x + y \to 4y = 2 + y \to 3y = 2$, which does not have a solution in $\mathbb{Z}$.

11 & 12. If $x = 1/2$, then the equation becomes $y = y + 1/2$, which has no solution in the domain.