Math 142 Exam I Solutions

Prof. Joshua Cooper, Spring 2010

Note that there are often multiple ways of solving problems, so you may still have a correct solution even if it does not agree with what is written here.

1. Evaluate the following integral:

$$\int \cos(\ln x)\, dx$$

Solution:
Let $u = \ln x$, so $du = \frac{1}{x}\, dx$, so

$$dx = x\, du = e^u du.$$

Then the integral becomes

$$\int e^u \cos u\, du.$$

This is computed using integration by parts *twice*. First $u_1 = e^u$, $dv_1 = \cos u\, du$, so $du_1 = e^u$ and $v_1 = \sin u$. Then

$$\int e^u \cos u\, du = e^u \sin u - \int e^u \sin u\, du.$$

Then $u_2 = e^u$, $dv_2 = \sin u\, du$, so $du_2 = e^u$ and $v_2 = -\cos u$, so

$$\begin{aligned}\int e^u \cos u\, du &= e^u \sin u - \int e^u \sin u\, du \\ &= e^u \sin u - \left(-e^u \cos u + \int e^u \cos u\, du\right) \\ &= e^u \sin u + e^u \cos u - \int e^u \cos u\, du.\end{aligned}$$

Adding $\int e^u \cos u\, du$ to both sides and dividing by 2 yields

$$\int e^u \cos u\, du = \frac{e^u}{2} \left(\sin u + \cos u\right).$$

Replacing $u = \ln x$, we have the final answer,

$$\int \cos(\ln x)\,dx = \frac{e^{\ln x}}{2}\left(\sin(\ln x) + \cos(\ln x)\right) = \frac{x}{2}\left(\sin(\ln x) + \cos(\ln x)\right).$$

2. Evaluate the following integral:

$$\int_0^{\pi/3} \tan^5 x \sec^3 x\,dx$$

Solution:
Make the substitution $u = \sec x$, so $du = \sec x \tan x\,dx$. Then

$$\tan^4 x = (\tan^2 x)^2 = (\sec^2 x - 1)^2 = (u^2 - 1)^2,$$

so, using the facts that $\sec 0 = 1$ and $\sec(\pi/3) = 2$, the integral becomes

$$\begin{aligned}
\int_1^2 (u^2 - 1)^2 u^2\,du &= \int_1^2 (u^4 - 2u^2 + 1)u^2\,du \\
&= \int_1^2 u^6 - 2u^4 + u^2\,du \\
&= \left.\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3}\right]_1^2 \\
&= \frac{2^7 - 1}{7} - \frac{2\cdot(2^5 - 1)}{5} + \frac{2^3 - 1}{3} \\
&= \frac{127}{7} - \frac{62}{5} + \frac{7}{3} \\
&= \frac{1905 - 1302 + 245}{105} = \frac{848}{105}.
\end{aligned}$$

3. Evaluate the following integral:

$$\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$$

Solution:
First, complete the square in the quadratic polynomial that appears in the denominator:

$$x^2 + 2x + 5 = (x + 1)^2 + 4.$$

Then we make the substitution $x+1 = 2\tan\theta$, so $dx = 2\sec^2\theta\, d\theta$, and

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+2x+5}} &= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\tan^2\theta+4}} \\
&= \int \frac{\sec^2\theta}{\sqrt{\tan^2\theta+1}}\, d\theta \\
&= \int \frac{\sec^2\theta}{\sec\theta}\, d\theta \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

If we solve $x+1 = 2\tan\theta$ for $\tan\theta$, we obtain

$$\tan\theta = \frac{x+1}{2}.$$

Furthermore, since $\sec\theta = \sqrt{\tan^2\theta+1}$, we have

$$\sec\theta = \sqrt{\frac{(x+1)^2}{4}+1} = \frac{1}{2}\sqrt{x^2+2x+5}.$$

(This can also be obtained by drawing the appropriate right triangle.) Therefore, putting the pieces together,

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+2x+5}} &= \ln\left|\frac{x+1}{2} + \frac{1}{2}\sqrt{x^2+2x+5}\right| + C \\
&= \ln\left|x+1+\sqrt{x^2+2x+5}\right| + C'
\end{aligned}$$

where $C' = C - \ln 2$.

4. Evaluate the following integral:

$$\int \frac{2x^3+3x+5}{(x-1)^2(x^2+4)}\, dx$$

Solution:
We proceed by partial fractions. The numerator has degree 3 and the denominator 4, so no long division is necessary. Then we write

$$\frac{2x^3+3x+5}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4}.$$

Multiplying both sides by $(x-1)^2(x^2+4)$,

$$2x^3 + 3x + 5 = A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2.$$

If we plug in $x = 1$, the resulting equation is

$$10 = 5B,$$

so $B = 2$. Now, equate the coefficients of cubic terms on both sides:

$$2 = A + C,$$

which we may rewrite as $C = 2 - A$. Equating the quadratic coefficients yields

$$0 = -A + B + D - 2C = -A + 2 + D - 2C$$

and equating the linear coefficients yields

$$3 = 4A + C - 2D$$

Substituting $C = 2 - A$ into the last two equations gives

$$A + D = 2 \qquad (1)$$

$$3A - 2D = 1. \qquad (2)$$

If we add twice the first equation to the second, the result is $5A = 5$, so $A = 1$. But then $C = 2 - A = 1$ and $D = 2 - A = 1$ (by equation (1)), so we may write the original integral as

$$\int \frac{2x^3 + 3x + 5}{(x-1)^2(x^2+4)}\,dx = \int \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{x+1}{x^2+4}\,dx.$$

We tackle this integral one summand at a time. First of all,

$$\int \frac{1}{x-1}\,dx = \ln|x-1| + C$$

and

$$\int \frac{2}{(x-1)^2}\,dx = -\frac{2}{x-1} + C.$$

Now, if we let $x = 2\tan\theta$ in the third summand, then $dx = 2\sec^2\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{x+1}{x^2+4}\, dx &= \int \frac{2\tan\theta + 1}{4\tan^2\theta + 4} \cdot 2\sec^2\theta\, d\theta \\
&= \frac{1}{2}\int \frac{(2\tan\theta + 1)(\sec^2\theta)}{\sec^2\theta}\, d\theta \\
&= \frac{1}{2}\int 2\tan\theta + 1\, d\theta \\
&= \frac{1}{2}\left(2\ln|\sec\theta| + \theta\right) + C \\
&= \ln|\sec\theta| + \frac{\theta}{2} + C.
\end{aligned}$$

Solving $x = 2\tan\theta$ for θ gives $\theta = \tan^{-1}(x/2)$. Also,

$$\sec\theta = \sqrt{\tan^2\theta + 1} = \sqrt{\frac{x^2}{4} + 1} = \frac{1}{2}\sqrt{x^2+4}.$$

Therefore,

$$\begin{aligned}
\int \frac{x+1}{x^2+4}\, dx &= \ln\left|\frac{1}{2}\sqrt{x^2+4}\right| + \frac{\tan^{-1}(x/2)}{2} + C \\
&= \frac{1}{2}\ln\left|x^2+4\right| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C'
\end{aligned}$$

where $C' = C - \ln 2$. Putting the pieces together,

$$\begin{aligned}
\int \frac{2x^3 + 3x + 5}{(x-1)^2(x^2+4)}\, dx =& \ln|x-1| - \frac{2}{x-1} \\
&+ \frac{1}{2}\ln\left|x^2+4\right| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.
\end{aligned}$$

5. Find the area of the region bounded by $x = 0$, $y = 0$, and

$$y = \frac{\sqrt{x} - x}{\sqrt{x} + x}$$

Solution:
First of all, note that the function $f(x) = (\sqrt{x} - x)/(\sqrt{x} + x)$ is only

defined for $x \geq 0$. Also, $f(x) = 0$ implies $\sqrt{x} = x$, which only occurs when $x = 0$ or $x = 1$. So we are really just computing the integral

$$\int_0^1 \frac{\sqrt{x} - x}{\sqrt{x} + x}\, dx.$$

Let $u = \sqrt{x}$, so that differentiating $u^2 = x$ gives $2u\,du = dx$. Then we may rewrite the indefinite integral as

$$\int \frac{u - u^2}{u + u^2} \cdot 2u\,du = 2\int \frac{u(1-u)}{1+u}\, du.$$

The degree of the polynomial in the numerator is greater than the one in the denominator, so we perform long division to obtain

$$\frac{u(1-u)}{1+u} = \frac{u - u^2}{1+u} = -u + 2 - \frac{2}{1+u},$$

so we may write

$$\begin{aligned}
\int_0^1 \frac{\sqrt{x} - x}{\sqrt{x} + x}\, dx &= 2\int_0^1 -u + 2 - \frac{2}{1+u}\, du \\
&= 2\left(-\frac{u^2}{2} + 2u - 2\ln|1+u| \Big]_0^1 \right) \\
&= 2\left(-\frac{1}{2} + 2 - 2\ln 2 - 2\ln 1 \right) = 3 - \ln 16,
\end{aligned}$$

where we have used the fact that $x = 0$ implies $u = 0$ and $x = 1$ implies $u = 1$.

6. For each of the following, indicate whether the improper integral is convergent or divergent by checking the appropriate box. Evaluate those which are convergent. (You do not need to show any work for

the divergent integrals.)

Integral	Convergent	Divergent	Value
$\int_1^\infty \frac{dx}{x}$	☐	☒	______
$\int_3^7 \frac{1}{(x-3)^3}\,dx$	☐	☒	______
$\int_{-\infty}^\infty xe^{-x}\,dx$	☐	☒	______
$\int_{-\infty}^{-2} \frac{\sqrt{x^2+1}}{x^{4/3}}\,dx$	☐	☒	______
$\int_0^5 x\ln x\,dx$	☒	☐	$\frac{25}{2}\ln 5 - \frac{25}{4}$
$\int_0^1 \frac{e^{1/x}}{x^2}\,dx$	☐	☒	______

Solution:

(a) $\int_1^\infty \frac{dx}{x}$

We covered in class that $\int_1^\infty x^p\,dx$ is divergent whenever $p \geq -1$. Here $p = -1$, so the integral is divergent.

(b) $\int_3^7 \frac{1}{(x-3)^3}\,dx$

Evaluate the integral directly.

$$\begin{aligned}\int_3^7 \frac{1}{(x-3)^3}\,dx &= \lim_{t\to 3+}\int_t^7 \frac{1}{(x-3)^3}\,dx \\ &= \lim_{t\to 3+}\left[-\frac{1}{2(x-3)^2}\right]_t^7 \\ &= \lim_{t\to 3+}\left[-\frac{1}{2(t-3)^2} + \frac{1}{32}\right]_t^7 \\ &= \infty,\end{aligned}$$

so the integral is divergent.

(c) $\displaystyle\int_{-\infty}^{\infty} xe^{-x}\,dx$

Because the integral has two improper limits, we have to split it:

$$\int_{-\infty}^{\infty} xe^{-x}\,dx = \int_{-\infty}^{0} xe^{-x}\,dx + \int_{0}^{\infty} xe^{-x}\,dx.$$

The function $f(x) = xe^{-x}$ has a finite integral on the interval $[0,\infty)$, but not the interval $(-\infty, 0]$. To see this, we first compute the indefinite integral by parts (taking $u = x$ and $dv = e^{-x}\,dx$):

$$\begin{aligned}\int xe^{-x}\,dx &= -xe^{-x} + \int e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} = (1-x)e^{-x}.\end{aligned}$$

Then the first definite (improper) integral is given by

$$\begin{aligned}\int_{-\infty}^{0} xe^{-x}\,dx &= \lim_{t\to-\infty} \int_{t}^{0} xe^{-x}\,dx \\ &= \lim_{t\to-\infty} \left[(1-x)e^{-x}\right]_t^0 \\ &= 1 + \lim_{t\to-\infty}(1-t)e^{-t}.\end{aligned}$$

However, $\lim_{t\to-\infty}(1-t) = \infty$ and $\lim_{t\to-\infty} e^{-t} = \infty$, so the above limit is ∞, and the integral is divergent. This implies that the original, two-sided integral is also divergent.

(d) $\displaystyle\int_{-\infty}^{-2} \frac{\sqrt{x^2+1}}{x^{4/3}}\,dx$

We use the Comparison Theorem. Note that, first of all, we can rewrite the integral as

$$\int_{-\infty}^{-2} \frac{\sqrt{x^2+1}}{x^{4/3}}\,dx = \int_{2}^{\infty} \frac{\sqrt{x^2+1}}{x^{4/3}}\,dx,$$

since the integrand is an even function. Then we use the fact that

$$\frac{\sqrt{x^2+1}}{x^{4/3}} > \frac{\sqrt{x^2}}{x^{4/3}} = \frac{x}{x^{4/3}} = x^{-1/3}.$$

Therefore, if $\int_2^\infty x^{-1/3}\,dx$ is divergent, then the original integral is as well. Again, we showed that $\int_1^\infty x^p\,dx$ is divergent whenever $p \geq -1$. Here $p = -1/3$, so the integral is divergent.

(e) $\displaystyle\int_0^5 x \ln x\,dx$

We integrate by parts, taking $u = \ln x$ and $dv = x\,dx$:

$$\begin{aligned}
\int_0^5 x \ln x\,dx &= \lim_{t\to 0+} \int_t^5 x \ln x\,dx \\
&= \lim_{t\to 0+} \left(\frac{x^2}{2} \ln x \Big]_t^5 - \int_t^5 \frac{x^2}{2} \cdot \frac{1}{x}\,dx \right) \\
&= \lim_{t\to 0+} \left(\frac{25}{2} \ln 5 - \frac{t^2}{2} \ln t - \int_t^5 \frac{x}{2}\,dx \right) \\
&= \frac{25}{2} \ln 5 - \lim_{t\to 0+} \left(\frac{t^2}{2} \ln t + \left[\frac{x^2}{4} \right]_t^5 \right) \\
&= \frac{25}{2} \ln 5 - \lim_{t\to 0+} \left(\frac{t^2}{2} \ln t + \frac{25}{4} - \frac{t^2}{4} \right) \\
&= \frac{25}{2} \ln 5 - \frac{25}{4} - \frac{1}{2} \lim_{t\to 0+} t^2 \ln t
\end{aligned}$$

So, it remains to evaluate the limit

$$\lim_{t\to 0+} t^2 \ln t = \lim_{t\to 0+} \frac{\ln t}{t^{-2}}.$$

This is a limit of the form ∞/∞, so we may apply L'Hôpital's Rule:

$$\begin{aligned}
\lim_{t\to 0+} \frac{\ln t}{t^{-2}} &= \lim_{t\to 0+} \frac{t^{-1}}{-2t^{-3}} \\
&= \lim_{t\to 0+} -\frac{t^2}{2} = 0.
\end{aligned}$$

Therefore, the integral converges, and its value is given by

$$\int_0^5 x \ln x\,dx = \lim_{t\to 0+} \int_t^5 x \ln x\,dx = \frac{25}{2} \ln 5 - \frac{25}{4}.$$

(f) $\displaystyle\int_0^1 \frac{e^{1/x}}{x^2}\,dx$

We again evaluate the integral directly.

$$\int_0^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0+} \int_t^1 \frac{e^{1/x}}{x^2}\,dx$$

Make the substitution $u = 1/x$. Then $1/u = x$, so $-du/u^2 = dx$, and

$$\begin{aligned}
\int_0^1 \frac{e^{1/x}}{x^2}\,dx &= -\lim_{t\to\infty} \int_t^1 \frac{e^u}{u^{-2}} \cdot u^{-2}\,du \\
&= \lim_{t\to\infty} \int_1^t e^u\,du \\
&= \lim_{t\to\infty} \left[e^u\right]_1^t \\
&= \lim_{t\to\infty} \left(e^t - e\right) \\
&= \infty,
\end{aligned}$$

so the integral is divergent.