

# Monochromatic Boxes in Colored Grids

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## Abstract

A  $d$ -dimensional *grid* is a set of the form  $R = [a_1] \times \cdots \times [a_d]$ . A  $d$ -dimensional *box* is a set of the form  $\{b_1, c_1\} \times \cdots \times \{b_d, c_d\}$ . When a grid is  $c$ -colored, must it admit a monochromatic box? If so, we say that  $R$  is  $c$ -guaranteed. This question is a relaxation of one attack on bounding the van der Waerden numbers, and also arises as a natural hypergraph Ramsey problem (viz. the Ramsey numbers of hyperoctahedra). We give conditions on the  $a_i$  for  $R$  to be  $c$ -guaranteed that are asymptotically tight, and analyze the set of minimally  $c$ -guaranteed grids.

## 1 Introduction

A  $d$ -dimensional *grid* is a set  $R = [a_1] \times \cdots \times [a_d]$ , where  $[t] = \{1, \dots, t\}$ . For ease of notation, we write  $[a_1, \dots, a_d]$  for  $[a_1] \times \cdots \times [a_d]$ . The “volume” of  $R$  is  $\prod_{i=1}^d a_i$ . A  $d$ -dimensional *box* is a set of  $2^d$  points of the form

$$\{(x_1 + \epsilon_1 s_1, \dots, x_d + \epsilon_d s_d) \mid \epsilon_i \in \{0, 1\} \text{ for } 1 \leq i \leq d\},$$

with  $s_i \neq 0$  for all  $1 \leq i \leq d$ . A grid  $R$  is  $(c, t)$ -*guaranteed*, if for all colorings  $f : R \rightarrow [c]$ , there are at least  $t$  distinct monochromatic boxes in  $R$ , i.e., boxes  $B_j \subseteq R$ ,  $j \in [t]$ , so that  $|f(B_j)| = 1$ . When  $t = 1$ , we simply say that  $R$  is  $c$ -guaranteed. If  $R$  is not  $c$ -guaranteed, we say it is  $c$ -*colorable*. Clearly, whether a grid is  $(c, t)$ -guaranteed depends only on  $a_1, \dots, a_d$ . Furthermore, if  $b_i \geq a_i$  for all  $i$  such that  $1 \leq i \leq d$ , then  $[b_1, \dots, b_d]$  is  $c$ -guaranteed if  $[a_1, \dots, a_d]$  is. This ordering on  $d$ -tuples is sometimes called the *dominance order*, and we will denote it by  $\preceq$ . Then one may state the above observation as the fact that the set of  $c$ -guaranteed grids is an up-set in the  $(\mathbb{N}^d, \preceq)$ -poset. Hence, we have a full understanding of this family if we know the minimal  $c$ -guaranteed grids, an antichain in the  $\preceq$  order. (Note that any such antichain is finite, a well-known fact in poset theory.) Call the set of minimal  $c$ -guaranteed grids  $\mathcal{O}(c, d)$ , the *obstruction set* for  $c$  colors in dimension  $d$ . We will focus our attention on *monotone* obstruction set elements, i.e., those grids for which  $a_1 \leq \cdots \leq a_d$ ,

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since being  $c$ -guaranteed (or  $(c, t)$ -guaranteed) is invariant under permutations of the  $a_j$ .

The subject of unavoidable configurations in grids has connections with the celebrated Van der Waerden's and Szemerédi's Theorems. (See, for example, [2], [5], and [8].) Our results can be seen as belonging to hypergraph Ramsey theory, as follows. Let  $G$  be the complete  $d$ -partite  $d$ -uniform hypergraph with blocks of size  $a_1, \dots, a_d$ . Then an edge of  $G$  can be identified with a vertex of  $R = [a_1, \dots, a_d]$  in the natural way. Under this correspondence, a  $c$ -coloring of  $R$  gives rise to a  $c$ -edge coloring of  $G$ , and boxes correspond precisely to subgraphs isomorphic to the "generalized octahedron"  $K_d(2)$ , the complete  $d$ -partite  $d$ -uniform hypergraph with each block of size 2. The generalized octahedra play an important and closely related role in the work of Kohayakawa, Rödl, and Skokan ([7]) on hypergraph quasirandomness. (Among other interesting results, they show that, asymptotically, a random  $c$ -edge coloring of  $G$  has the fewest number of monochromatic  $K_d(2)$ 's possible.) We may translate each of our results into statements about the Ramsey numbers of hyperoctahedra-free  $d$ -partite  $d$ -uniform graphs. For example, in Section 7, we give a family of upper bounds on the sizes of 3-dimensional grids which have a 2-coloring admitting no monochromatic box; this is equivalent to asking for the extremal tripartite 3-uniform hypergraphs which are  $(K_3(2), K_3(2))$ -Ramsey.

The present work is even more closely connected to the "Product Ramsey Theorem." Though the proof appears in [6], the statement appearing in [9] best illustrates the connection:

**Theorem 1.1** (Product Ramsey Theorem). *Let  $k_1, \dots, k_d$  be nonnegative integers; let  $c$  and  $d$  be positive integers; and let  $m_1, \dots, m_d$  be integers with  $m_i \geq k_i$  for  $i \in [d]$ . Then there exists an integer  $R = R(c, d; k_1, \dots, k_d; m_1, \dots, m_d)$  so that if  $X_1, \dots, X_d$  are sets and  $|X_i| \geq R$  for  $i \in [d]$ , then for every function  $f : \binom{X_1}{k_1} \times \dots \times \binom{X_d}{k_d} \rightarrow [c]$ , there exists an element  $\alpha \in [c]$  and subsets  $Y_1, \dots, Y_d$  of  $X_1, \dots, X_d$ , respectively, so that  $|Y_i| \geq m_i$  for  $i \in [d]$  and  $f$  maps every element of  $\binom{X_1}{k_1} \times \dots \times \binom{X_d}{k_d}$  to  $\alpha$ .*

This result ensures that the quantity  $N(c, d) = R(c, d; 1^d, 2^d)$ , which corresponds to the least  $R$  so that  $[R]^d$  is  $c$ -guaranteed, is finite. A closer analysis of  $N(c, d)$  – in fact, the more general  $N(c, d, m) = R(c, d; 1^d, m^d)$  – appears in the manuscript [1] by Agnarsson, Doerr, and Schoen. They obtain asymptotic bounds on  $N(c, d, m)$  that are valid for large  $m$ . Here, we examine instead the least nontrivial case of  $m = 2$ , and consider grids which are not necessarily equilateral.

In the next section, we show that any grid of sufficiently small volume (approximately  $c^{2^d - 1}$ ) is  $c$ -colorable. The following section shows that the analysis is tight: there are grids of this volume which are  $c$ -guaranteed. Not all grids of sufficient volume are  $c$ -guaranteed, although Section 4 demonstrates that any grid all of whose lower-dimensional subgrids are sufficiently voluminous is indeed  $c$ -guaranteed. The next section gives a tight upper bound on the volume of minimally  $c$ -guaranteed grids, i.e., elements of the obstruction set. Section

6 then addresses the question of how many obstructions there are. Finally, as mentioned above, Section 7 considers the case of  $c = 2$  and  $d = 3$ , where some interesting computational questions arise. This extends work of the second two authors ([4]) for  $d = 2$  and  $2 \leq c \leq 4$ .

Throughout the present manuscript, unless we explicitly say otherwise, we use the notations  $x = O(y)$  and  $y = \Omega(x)$  to mean that there is a function  $F : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  such that  $x \leq F(d)y$ . That is,  $x$  is bounded by  $y$  times a number that only depends on  $d$ . (Naturally,  $x = \Theta(y)$  means that  $x = O(y)$  and  $x = \Omega(y)$ , and notation  $x = o(y)$  is defined analogously.) In general,  $x$  and  $y$  will depend on  $c$ ,  $d$ , and perhaps other quantities.

## 2 All small grids are $c$ -colorable

Define  $V(c, d)$  to be the largest integer  $V$  so that every  $d$ -dimensional grid  $R$  with volume at most  $V$  is  $c$ -colorable. Below, we show that  $V(c, d)$  is  $\Theta(c^{2^d-1})$ .

**Theorem 2.1.**

$$V(c, d) = \Omega(c^{2^d-1}).$$

*In fact,  $V(c, d) > c^{2^d-1}/e^{2^d}$ , where  $e = 2.718\dots$  is Napier's constant.*

*Proof.* We apply the Lovász Local Lemma (see, e.g., [3]), which states the following. Suppose that  $A_1, \dots, A_t$  are events in some probability space, each of probability at most  $p$ . Let  $G$  be a “dependency” graph with vertex set  $\{A_i\}_{i=1}^t$ , i.e., a graph so that, whenever a set  $S$  of vertices induces no edges in  $G$ , then  $S$  is a mutually independent family of events. Then  $\mathbf{P}(\bigwedge_{i=1}^t \bar{A}_i) > 0$  if  $ep(\Delta+1) \leq 1$ , where  $\Delta = \Delta(G)$  is the maximum degree of  $G$ .

Now, suppose  $R = [a_1, \dots, a_d]$  is a grid of volume  $V$ , and we color the points of  $R$  uniformly at random from  $[c]$ . Enumerate all boxes in  $R$  as  $B_1, \dots, B_t$ . Define  $A_i$  to be the event that  $B_t$  is monochromatic in this random coloring. Clearly, we may take  $G$  to have an edge between  $A_i$  and  $A_j$  whenever  $B_i \cap B_j \neq \emptyset$ . The degree of a vertex  $A_i$  is then the number of boxes  $B_j$ ,  $j \neq i$ , which intersect  $B_i$ . Since we may specify the list of all such boxes by choosing one of the  $2^d$  points of  $B_i$ , and then choosing the  $d$  coordinates of its antipodal point,  $\deg_G(A_i)$  is at most

$$2^d \prod_{i=1}^d (a_i - 1) - 1 < 2^d \prod_{i=1}^d a_i - 1 = 2^d V - 1.$$

(The outermost  $-1$  here reflects the fact that  $B_i$  may be excluded among these choices.) The probability of each  $A_i$  is the same:  $p = c^{-2^d+1}$ . Therefore,

$$ep(\Delta + 1) < ec^{-2^d+1}2^dV$$

which is  $\leq 1$  whenever  $V \leq c^{2^d-1}/e^{2^d}$ . □

### 3 Some large grids are $c$ -guaranteed

**Theorem 3.1.** Fix  $c, d$ , define  $R = [a_1, \dots, a_d]$ , and let  $M = \prod_i \binom{a_i}{2}$  denote the total number of boxes in  $R$ . For  $\min\{a_1, \dots, a_d\} \rightarrow \infty$ ,  $R$  is  $(c, M(1 + o(1))/c^{2^d-1})$ -guaranteed.

Theorem 3.1 follows quickly from the next lemma, whose extra strength we will need later.

**Lemma 3.2.** Suppose  $c \geq 1$ . For  $d \geq 1$  and integers  $a_1, \dots, a_d \geq 2$ , let  $M = \prod_{i=1}^d \binom{a_i}{2}$ . The grid  $R = [a_1, \dots, a_d]$  is  $(c, M\Delta_d/c^{2^d-1})$ -guaranteed provided  $\Delta_1, \dots, \Delta_d > 0$ , where  $\Delta_j$ ,  $0 \leq j \leq d$ , is given by the recurrence

$$\begin{aligned} \Delta_0 &= 1, \\ \Delta_j &= \Delta_{j-1}^2 \left( 1 - \frac{c^{2^{j-1}} - 1}{a_j - 1} \right). \end{aligned}$$

*Proof.* We proceed inductively. Suppose  $d = 1$ , let  $f : [a_1] \rightarrow [c]$  be a  $c$ -coloring, and define

$$\gamma_i = |f^{-1}(i)|$$

to be the number of points colored  $i$ ,  $1 \leq i \leq c$ . Then the number  $N$  of monochromatic boxes in  $f$  is exactly

$$N = \sum_{i=1}^c \binom{\gamma_i}{2} = \frac{1}{2} \cdot \sum_{i=1}^c (\gamma_i^2 - \gamma_i) = \frac{1}{2} \cdot \left( \sum_{i=1}^c \gamma_i^2 - a_1 \right).$$

Applying Cauchy-Schwarz,

$$\begin{aligned} N &\geq \frac{(\sum_{i=1}^c \gamma_i)^2}{2c} - \frac{a_1}{2} = \frac{a_1^2}{2c} - \frac{a_1}{2} \\ &= \frac{a_1(a_1 - c)}{2c} = \frac{1}{c} \binom{a_1}{2} \frac{a_1 - c}{a_1 - 1} = \frac{1}{c} \binom{a_1}{2} \Delta_1. \end{aligned}$$

Now, suppose the statement is true for dimensions  $< d + 1$ , and consider a coloring  $f : [a_1, \dots, a_{d+1}] \rightarrow [c]$ . Consider the  $a_{d+1}$  colorings  $f_j$  of the  $d$ -dimensional grid  $[a_1, \dots, a_d]$  induced by setting the last coordinate to  $j$ , i.e.,

$$f_j(x_1, \dots, x_d) = f(x_1, \dots, x_d, j).$$

Let  $\gamma_i(B)$ , for a box  $B \subset [a_1, \dots, a_d]$  and  $i \in [c]$ , denote the number of  $j$  so that  $f_j|_B \equiv i$ . Then the number  $N$  of monochromatic  $(d + 1)$ -dimensional boxes in  $f$  is

$$N = \sum_i \sum_B \binom{\gamma_i(B)}{2}$$

$$\begin{aligned}
&= \frac{1}{2} \cdot \sum_i \sum_B (\gamma_i(B)^2 - \gamma_i(B)) \\
&\geq \frac{(\sum_B \sum_i \gamma_i(B))^2}{2Mc} - \frac{1}{2} \cdot \sum_B \sum_i \gamma_i(B) \\
&= \frac{(\sum_B \sum_i \gamma_i(B))^2 - Mc \sum_B \sum_i \gamma_i(B)}{2Mc}
\end{aligned}$$

where  $M = \prod_{i=1}^d \binom{a_i}{2}$ . Since, by the inductive hypothesis,  $f_j$  induces at least  $M\Delta_d/c^{2^d-1}$  monochromatic boxes,

$$\sum_i \sum_B \gamma_i(B) \geq \frac{a_{d+1}M\Delta_d}{c^{2^d-1}},$$

so that

$$\begin{aligned}
N &\geq \frac{a_{d+1}^2 M^2 \Delta_d^2 / c^{2^{d+1}-2} - a_{d+1} M^2 \Delta_d c / c^{2^d-1}}{2Mc} \\
&= \frac{a_{d+1} M (a_{d+1} \Delta_d^2 - c^{2^d} \Delta_d)}{2c^{2^{d+1}-1}} \\
&= \frac{M}{c^{2^{d+1}-1}} \binom{a_{d+1}}{2} \frac{a_{d+1} \Delta_d^2 - c^{2^d} \Delta_d}{a_{d+1} - 1} \\
&= \frac{\prod_{i=1}^{d+1} \binom{a_i}{2}}{c^{2^{d+1}-1}} \cdot \Delta_d^2 \left( \frac{a_{d+1} - c^{2^d} / \Delta_d}{a_{d+1} - 1} \right) \\
&= \frac{\prod_{i=1}^{d+1} \binom{a_i}{2} \Delta_{d+1}}{c^{2^{d+1}-1}}.
\end{aligned}$$

□

*Proof of Theorem 3.1.* Fix  $c, d \geq 1$ . It is clear by induction on  $j$  that for all  $1 \leq j \leq d$ , as  $\min\{a_1, \dots, a_d\} \rightarrow \infty$ ,  $\Delta_j = 1 + o(1)$ , and so in particular,  $\Delta_j > 0$  if  $\min\{a_1, \dots, a_d\}$  is large enough. □

Note that, in the notation of Lemma 3.2, if  $\Delta_1, \dots, \Delta_d > 0$ , then  $[a_1, \dots, a_d]$  is not  $c$ -colorable. Therefore we may conclude the following.

**Corollary 3.3.** *In the notation of Lemma 3.2, let  $\Gamma_j$ ,  $0 \leq j \leq d$ , be given by the recurrence*

$$\begin{aligned}
\Gamma_0 &= 1, \\
\Gamma_j &= \Gamma_{j-1}^2 \left( 1 - \frac{c^{2^{j-1}} / \Gamma_{j-1}}{a_j - 1} \right) = \Gamma_{j-1} \left( \Gamma_{j-1} - \frac{c^{2^{j-1}}}{a_j - 1} \right).
\end{aligned}$$

*If  $\Gamma_1, \dots, \Gamma_d > 0$ , then  $[a_1, \dots, a_d]$  is  $c$ -guaranteed.*

*Proof.* Assume  $\Gamma_1, \dots, \Gamma_d > 0$ . A routine induction shows that  $\Gamma_j \leq \Delta_j$  for  $0 \leq j \leq d$ .  $\square$

**Lemma 3.4.** *In the notation of Lemma 3.2, let  $\varepsilon_j$  be given by the recurrence*

$$\begin{aligned}\varepsilon_0 &= 0, \\ \varepsilon_j &= 2\varepsilon_{j-1} + \frac{c^{2^{j-1}}}{a_j - 1}.\end{aligned}$$

*If  $\varepsilon_d < 1$ , then  $[a_1, \dots, a_d]$  is  $c$ -guaranteed.*

*Proof.* Clearly,  $0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \dots < \varepsilon_d$ , and so by assumption  $\varepsilon_i < 1$  for all  $i \in [d]$ . An induction on  $i$  shows that  $\Gamma_i \geq 1 - \varepsilon_i$  for  $0 \leq i \leq d$ : This is clearly true for  $i = 0$ . Suppose  $i < d$  and  $\Gamma_i \geq 1 - \varepsilon_i$ . Then setting  $\eta := c^{2^i}/(a_{i+1} - 1)$  and noting that  $\Gamma_i \geq 0$ , we have

$$\Gamma_{i+1} = \Gamma_i(\Gamma_i - \eta) \geq \Gamma_i(1 - \varepsilon_i - \eta). \quad (1)$$

The term in the parentheses is positive:

$$1 - \varepsilon_i - \eta \geq 1 - 2\varepsilon_i - \eta = 1 - \varepsilon_{i+1} > 0$$

by assumption. Thus continuing (1) and using the inductive hypothesis again,

$$\Gamma_i(1 - \varepsilon_i - \eta) \geq (1 - \varepsilon_i)(1 - \varepsilon_i - \eta) \geq 1 - 2\varepsilon_i - \eta = 1 - \varepsilon_{i+1}.$$

$\square$

Motivated by the preceding lemma, for every  $c, d \geq 1$  and grid  $R = [a_1, \dots, a_d]$  with  $a_i \geq 2$  for all  $i \in [d]$ , we define

$$\varepsilon_c(R) := \sum_{i=1}^d 2^{d-i} \frac{c^{2^{i-1}}}{a_i - 1}.$$

**Lemma 3.5.** *If  $R = [a_1, \dots, a_d]$  is not  $c$ -guaranteed, then  $\varepsilon_c(R) \geq 1$ .*

*Proof.* We let  $\varepsilon_j := \varepsilon_c([a_1, \dots, a_j]) = \sum_{i=1}^j 2^{j-i} c^{2^{i-1}}/(a_i - 1)$  for all  $j$  with  $0 \leq j \leq d$ , and notice that the  $\varepsilon_j$  satisfy the recurrence in Lemma 3.4.  $\square$

**Corollary 3.6.** *For any fixed  $d \geq 1$  and  $c \geq 2$ , if  $n$  is least such that  $[n]^d$  is  $c$ -guaranteed, then  $n < (d+2)c^{2^{d-1}}$ . Furthermore,*

$$2^{-d}e^{-1} < \frac{V(c, d)}{c^{2^d-1}} < (d+2)^d 2^{d(d-1)/2}.$$

*Proof.* If we take  $a_j = (d+1)2^{d-j}c^{2^{j-1}} + 1$  for all  $1 \leq j \leq d$ , then  $\varepsilon_c(R) = d/(d+1) < 1$ . The second result now follows from the fact that

$$\prod_{j=1}^d a_j < \prod_{j=1}^d (d+2)2^{d-j}c^{2^{j-1}}$$

$$\begin{aligned}
&= (d+2)^{d \sum_{j=1}^d (d-j)} c^{\sum_{j=1}^d 2^{j-1}} \\
&= (d+2)^{d \sum_{j=1}^{d-1} j} c^{\sum_{j=0}^{d-1} 2^j} \\
&= (d+2)^{d \cdot 2^{d-1}/2} c^{2^d - 1}.
\end{aligned}$$

The first result follows by taking  $n := a_d$ .  $\square$

## 4 Hereditarily large grids are $c$ -guaranteed

It is possible for grids of arbitrarily large volume to be  $c$ -colorable. Indeed, one need only have one of the dimensions be at most  $c$ , and then color the grid with this coordinate. However, if we require that each lower dimensional sub-grid be sufficiently voluminous, then the whole grid is  $c$ -colorable. This statement is made precise by the following theorem.

**Theorem 4.1.** *Fix  $d > 0$ , and define  $C_j = (d2^d)^{\frac{3}{2}(3^{j-1}-1)}$  for  $j \geq 1$ . For all integers  $c \geq 1$  and  $1 \leq a_1 \leq a_2 \leq \dots \leq a_d$ , if  $\prod_{i=1}^j a_i > C_j c^{(3^j-1)/2}$  for all  $j \in [d]$ , then  $[a_1, \dots, a_d]$  is  $c$ -guaranteed.*

We require a lemma and a bit of notation: If  $R = [a_1, \dots, a_d]$  and  $1 \leq j < d$ , let  $R_j$  denote  $[a_1, \dots, a_j]$  and let  $\overline{R}_j$  denote  $[a_{j+1}, \dots, a_d]$ . Note that, if  $R$  is  $c$ -guaranteed, then  $R_j$  is as well. Indeed, if  $f : R_j \rightarrow [c]$  is a  $c$ -coloring of  $R_j$ , then the function  $g : R \rightarrow [c]$  defined by  $g(x_1, \dots, x_d) = f(x_1, \dots, x_j)$  is a  $c$ -coloring of  $R$ . We will also make repeated use of the following easily verified fact: For every integer  $j \geq 0$ ,  $j \cdot 2^{j-1} \leq (3^j - 1)/2$  and  $j \cdot 2^j + 1 \leq 3^j$ .

**Lemma 4.2.** *Let  $c \geq 1$ , let  $R = [a_1, \dots, a_d]$  be a grid, and let  $j \in [d-1]$ . Define*

$$c' := c \cdot \prod_{i=1}^j \binom{a_i}{2} \leq 2^{-j} \cdot c \cdot \prod_{i=1}^j a_i^2.$$

*If  $R_j$  is  $c$ -guaranteed and  $\overline{R}_j$  is  $c'$ -guaranteed, then  $R$  is  $c$ -guaranteed.*

*Proof.* Assume that  $R_j$  is  $c$ -guaranteed and that  $\overline{R}_j$  is  $c'$ -guaranteed. Suppose that  $f : R \rightarrow [c]$  is a  $c$ -coloring. Consider the coloring  $g : \overline{R}_j \rightarrow [c']$  that assigns the pair  $(B, s)$  to the point  $\mathbf{v}$ ,  $B$  being an arbitrary choice of  $j$ -dimensional box colored monochromatically by  $f_j : R_j \rightarrow [c]$ , where  $f_j(x_1, \dots, x_j) = f(x_1, \dots, x_j, \mathbf{v})$ , and  $s$  being its color. (Note that  $R_j$  is  $c$ -guaranteed, so such a  $B$  always exists.) Then  $g$  is a  $c'$ -coloring, because there are exactly  $c'$  many different  $(B, s)$ . Since  $\overline{R}_j$  is  $c'$ -guaranteed,  $g$  colors some  $(d-j)$ -dimensional box  $B_1$  monochromatically, with color  $(B_2, s)$ . But then  $B_2 \times B_1$  is a  $d$ -dimensional box monocolored by  $f$  with color  $s$ .  $\square$

*Proof of Theorem 4.1.* The statement is clearly true when  $d = 1$  since  $C_1 = 1$ . Suppose  $d > 1$  and the statement is true for all  $d' < d$ . Let  $R = [a_1, \dots, a_d]$  be a monotone grid satisfying the hypothesis of the theorem.

*Case 1:*  $\varepsilon_c(R) < 1$ . The result follows immediately from Lemma 3.5.

Case 2:  $\varepsilon_c(R) \geq 1$ . Then there is some  $j \in [d]$  such that  $2^{d-j}c^{2^{j-1}}/(a_j-1) \geq 1/d$ , i.e.,

$$a_j \leq d2^{d-j}c^{2^{j-1}} + 1 < d2^{d-j+1}c^{2^{j-1}}.$$

Since  $j2^j \leq 3^j - 1$  for all integers  $j \geq 1$ ,

$$\prod_{i=1}^j a_i \leq \prod_{i=1}^j a_j < d^j 2^{j(d-j+1)} c^{j2^{j-1}} \leq d^j 2^{j(d-j+1)} c^{(3^j-1)/2},$$

and so for all  $k \in [d-j]$ ,

$$\prod_{i=1}^k a_{j+i} > \frac{C_{j+k}}{d^j 2^{j(d-j+1)}} c^{(3^{j+k}-1)/2 - (3^j-1)/2} \geq \frac{C_{j+k}}{d^j 2^{j(d-j+1)}} c^{3^j(3^k-1)/2}.$$

Let  $c' = d^{2j} 2^{2j(d-j+1)} c^{3^j}$ . (Note that  $c' \geq c \cdot \prod_{i=1}^j a_i^2$ .) Then for all  $k \in [d-j]$ ,

$$\begin{aligned} \prod_{i=1}^k a_{j+i} &> \frac{C_{j+k}}{d^j 2^{j(d-j+1)}} \left( \frac{c'}{d^{2j} 2^{2j(d-j+1)}} \right)^{(3^k-1)/2} \\ &= \frac{(d2^d)^{\frac{3}{2}(3^{j+k-1}-1)}}{(d2^{d-j+1})j3^k} c'^{(3^k-1)/2} \\ &\geq (d2^d)^{\frac{3}{2}(3^{j+k-1}-1) - j3^k} c'^{(3^k-1)/2} \\ &\geq (d2^d)^{\frac{3}{2}(3^{j+k-1}-1) - (3^j-1)3^k/2} c'^{(3^k-1)/2}, \end{aligned}$$

because  $j \leq (3^j - 1)/2$  for all  $j \geq 1$ . Continuing the computation,

$$\begin{aligned} \prod_{i=1}^k a_{j+i} &> (d2^d)^{\frac{3}{2}(3^{j+k-1}-1) - (3^j-1)3^k/2} c'^{(3^k-1)/2} \\ &= (d2^d)^{\frac{3}{2}(3^{j+k-1}-1-3^{j+k-1}+3^k-1)} c'^{(3^k-1)/2} \\ &= (d2^d)^{\frac{3}{2}(3^k-1)} c'^{(3^k-1)/2} \\ &= C_k c'^{(3^k-1)/2}. \end{aligned}$$

Therefore  $\overline{R}_j = [a_{j+1}, \dots, a_d]$  is  $c'$ -guaranteed by the inductive hypothesis. (It is easy to see that the  $C_j$ 's are increasing in  $d$ , so taking  $d' = d - j$  causes no problem here.) Since  $R_j$  is also  $c$ -guaranteed by the inductive hypothesis, we may apply Lemma 4.2 to conclude that  $R$  is  $c$ -guaranteed.  $\square$

## 5 Upper bounds on the volume of obstruction grids

Before proceeding, we introduce the following notation. For  $d \geq 1$  and any monotone grid  $R = [a_1, \dots, a_d]$  where  $a_d > 1$ , we let  $R^-$  denote the monotone

grid obtained from  $R$  by subtracting one from  $a_j$ , where  $j \in [d]$  is least such that  $a_j = a_d$ . Note that if  $R$  is monotone and  $R \in \mathcal{O}(c, d)$ , then  $R$  is  $c$ -guaranteed but  $R^-$  is not  $c$ -guaranteed.

The next theorem gives an asymptotic upper bound on the volume  $\prod_{i=1}^d a_i$  of any grid  $[a_1, \dots, a_d] \in \mathcal{O}(c, d)$ .

**Theorem 5.1.** *For every  $d \geq 1$  and every grid  $R = [a_1, \dots, a_d] \in \mathcal{O}(c, d)$ ,*

$$\prod_{i=1}^d a_i = O\left(c^{(3^d-1)/2}\right).$$

The theorem follows immediately from the following lemma:

**Lemma 5.2.** *For every  $d \geq 1$ , every  $c \geq 2$ , and every monotone grid  $R = [a_1, \dots, a_d] \in \mathcal{O}(c, d)$ , there is a set  $P \subseteq [d]$  such that*

1.  $d \in P$ ,
2.  $\prod_{i=1}^{\ell} a_i = O\left(c^{(3^{\ell}-1)/2}\right)$  for every  $\ell \in P$ , and
3. For every  $k \in [d]$ ,

$$a_k = O\left(c^{3^j \cdot 2^{\ell-j-1}}\right),$$

where  $\ell$  is the least element of  $P$  that is  $\geq k$ , and  $j$  is the biggest element of  $P$  that is  $< k$  ( $j = 0$  if there is no such element).

(We call the elements of  $P$  pinch points for  $R$ .)

*Proof.* Let  $d \geq 1$  and  $c \geq 2$  be given, and let  $R = [a_1, \dots, a_d] \in \mathcal{O}(c, d)$  be a monotone grid. Then  $R$  is  $c$ -guaranteed, and thus  $R_j$  is also  $c$ -guaranteed for all  $1 \leq j \leq d$ . Since  $R \in \mathcal{O}(c, d)$ , we have that  $R^-$  is not  $c$ -guaranteed, and thus  $\varepsilon_c(R^-) \geq 1$ . This in turn implies that there is some largest  $\ell \in [d]$  such that

$$2^{d-\ell} \frac{c^{2^{\ell-1}}}{a_{\ell} - 2} \geq \frac{1}{d}.$$

(Note that the denominator is positive, because  $a_{\ell} \geq a_1 \geq c + 1 \geq 3$  since  $R$  is  $c$ -guaranteed.) Thus,

$$a_{\ell} \leq d2^{d-\ell} \cdot c^{2^{\ell-1}} + 2 \leq (d+2)2^{d-\ell} \cdot c^{2^{\ell-1}}, \quad (2)$$

and thus

$$\prod_{i=1}^{\ell} a_i \leq (a_{\ell})^{\ell} \leq ((d+2)2^{d-\ell})^{\ell} \cdot c^{\ell \cdot 2^{\ell-1}} \leq ((d+2)2^{d-1})^d \cdot c^{(3^{\ell}-1)/2}, \quad (3)$$

which implies that  $\ell$  satisfies Condition 2 of the lemma. We will make  $\ell$  the least element of  $P$ , noticing that Equation (2) and the monotonicity of  $R$  imply that  $a_k$  satisfies Condition 3 of the lemma for all  $k \in [\ell]$  (with  $j = 0$ ).

If  $\ell = d$ , then we let  $P = \{\ell\} = \{d\}$  and we are done.

Otherwise,  $\ell < d$ . Note that  $R^- = R_\ell \times (\overline{R}_\ell)^-$  up to a possible permutation of the coordinates. Recall also that  $R_\ell$  is  $c$ -guaranteed, but  $R^-$  is not. It follows from Lemma 4.2 that  $(\overline{R}_\ell)^-$  is not  $c'$ -guaranteed, where

$$c' := c \cdot \prod_{i=1}^{\ell} \binom{a_i}{2} = O \left( c \cdot \left( \prod_{i=1}^{\ell} a_i \right)^2 \right).$$

The bound in Equation (3) gives  $c' = O(c^{3^\ell})$ .

We thus have  $\varepsilon_{c'}((\overline{R}_\ell)^-) \geq 1$ , and so there is some largest  $m$  with  $\ell < m \leq d$  such that

$$2^{d-m} \frac{(c')^{2^{m-\ell-1}}}{a_m - 2} \geq \frac{1}{d - \ell},$$

which gives

$$a_m \leq (d - \ell) 2^{d-m} \cdot (c')^{2^{m-\ell-1}} + 2 \tag{4}$$

$$\leq (d - \ell + 2) 2^{d-m} \cdot (c')^{2^{m-\ell-1}} \tag{5}$$

$$= O(c^{3^\ell \cdot 2^{m-\ell-1}}). \tag{6}$$

For the volume of  $R_m$ , we get

$$\begin{aligned} \prod_{i=1}^m a_i &= \prod_{i=1}^{\ell} a_i \cdot \prod_{i=\ell+1}^m a_i \\ &\leq \left( \prod_{i=1}^{\ell} a_i \right) \cdot (a_m)^{m-\ell} \\ &= O(c^{(3^\ell-1)/2}) \cdot O(c^{3^\ell \cdot (m-\ell) \cdot 2^{m-\ell-1}}) \\ &= O(c^{(3^\ell-1)/2} \cdot c^{3^\ell \cdot (3^m - 1)/2}) \\ &= O(c^{(3^m-1)/2}). \end{aligned}$$

We make  $\ell$  and  $m$  the two least elements of  $P$ , and the last calculation shows that  $m \in P$  satisfies Condition 2. Further, since  $a_k \leq a_m$  for all  $k$  such that  $\ell < k \leq m$ , Condition 3 is also satisfied for all these  $a_k$  by Equations (4)–(6).

If  $m = d$ , then we let  $P = \{\ell, m\}$  and we are done. Otherwise, we repeat the argument above using  $m$  instead of  $\ell$  to obtain an  $n$  with  $m < n \leq d$  such that  $\ell$ ,  $m$ , and  $n$  being the least three elements of  $P$  satisfies Conditions 2 and 3 of the lemma, and so on until we arrive at  $d$ , whence we set  $P := \{\ell, m, n, \dots, d\}$ .  $\square$

The next proposition shows that the bounds in Lemma 5.2 are asymptotically tight.

**Proposition 5.3.** For  $c \geq 2$ , there is an infinite sequence  $\{\mu_j(c)\}_{j=1}^\infty$  of positive integers such that

1.  $\mu_j(c) \geq 1 + 2^{(1-3^{j-1})/2} \cdot c^{3^{j-1}}$  for all  $j \in \mathbb{Z}^+$ , and
2. for all  $d \geq 1$ , the grid  $[\mu_1(c), \dots, \mu_d(c)] \in \mathcal{O}(c, d)$  with pinch point set  $P = [d]$ .

*Proof.* For all  $c \geq 2$ , define

$$\begin{aligned} \mu_1(c) &:= 1 + c, \\ \mu_2(c) &:= 1 + c \cdot \binom{c+1}{2}, \\ &\vdots \\ \mu_{j+1}(c) &:= 1 + c \cdot \prod_{i=1}^j \binom{\mu_i(c)}{2}, \\ &\vdots \end{aligned}$$

Fix  $c \geq 2$  and let  $\mu_j$  denote  $\mu_j(c)$  for short. A routine induction on  $j$  shows (1). For the inductive step, noting that  $\sum_{i=0}^{j-1} 3^i = (3^j - 1)/2$ , we have

$$\begin{aligned} \mu_{j+1} &= 1 + c \cdot \prod_{i=1}^j \binom{\mu_i}{2} \\ &\geq 1 + \frac{c}{2^j} \prod_{i=1}^j (\mu_i - 1)^2 \\ &\geq 1 + \frac{c}{2^j} \prod_{i=1}^j \frac{c^{2 \cdot 3^{i-1}}}{2^{3^i - 1} - 1} \\ &= 1 + \frac{c^{3^j}}{2^{(3^j - 1)/2}}. \end{aligned}$$

For (2), we use induction on  $d \geq 1$  to show separately that

1.  $[\mu_1, \dots, \mu_d]$  is  $c$ -guaranteed, and
2.  $[\mu_1, \dots, \mu_d]$  is not  $(c, 2)$ -guaranteed (i.e., there is a coloring  $[\mu_1, \dots, \mu_d] \rightarrow [c]$  that monochromes exactly one box).

Clearly  $[\mu_1] = [1 + c]$  is  $c$ -guaranteed by the Pigeonhole Principle. Now let  $d \geq 2$  and assume that  $[\mu_1, \dots, \mu_{d-1}]$  is  $c$ -guaranteed. Then letting  $c' = c \cdot \prod_{i=1}^{d-1} \binom{\mu_i}{2}$ , we have  $\mu_d = 1 + c'$ , and hence  $[\mu_d]$  is  $c'$ -guaranteed. But then,  $[\mu_1, \dots, \mu_d]$  is  $c$ -guaranteed by Lemma 4.2 (letting  $j = d - 1$ ).

Now for claim (2). For  $d = 1$ , clearly the coloring  $[\mu_1] \rightarrow [c]$  mapping  $j \mapsto (j \bmod c) + 1$  has exactly one monochromatic 1-dimensional box, namely,

$(1; c) = \{1, c + 1\}$ . Now let  $d \geq 2$  and assume claim (2) holds for  $d - 1$ , i.e., there is a coloring  $[\mu_1, \dots, \mu_{d-1}] \rightarrow [c]$  that monocolors exactly one box. We will call such a coloring *minimal*. This generates exactly  $\prod_{i=1}^{d-1} \binom{\mu_i}{2}$  many boxes in  $[\mu_1, \dots, \mu_{d-1}]$ . For each of these boxes  $B$  and for each color  $s$ , we can find a minimal coloring that monocolors  $B$  with  $s$  by permuting the order of the hyperplanes along each axis and by permuting the colors. Thus there are exactly  $c' = c \cdot \prod_{i=1}^{d-1} \binom{\mu_i}{2}$  many distinct minimal colorings. We overlay these  $c'$  many colorings to obtain a coloring of  $[\mu_1, \dots, \mu_{d-1}, c']$  with no monochromatic  $d$ -boxes. We then duplicate the first  $(d - 1)$ -dimensional layer to arrive at a  $c$ -coloring of  $[\mu_1, \dots, \mu_{d-1}, 1 + c'] = [\mu_1, \dots, \mu_d]$ . This coloring has only one monocolored  $d$ -box: the box corresponding to the duplicated layer of unique monocolored  $(d - 1)$ -boxes. This shows Item (2).

It follows from claim (2) that  $[\mu_1, \dots, \mu_{d-1}, \mu_d - 1]$  is not  $c$ -guaranteed for any  $d \geq 1$ , since we can remove a single hyperplane from the only monocolored  $d$ -box in some minimal coloring of  $[\mu_1, \dots, \mu_{d-1}, \mu_d]$  to leave a coloring of  $[\mu_1, \dots, \mu_{d-1}, \mu_d - 1]$  without any monochromatic  $(d - 1)$ -boxes. From this it easily follows that  $[\mu_1, \dots, \mu_d] \in \mathcal{O}(c, d)$ , because  $[\mu_1, \dots, \mu_{j-1}, \mu_j - 1]$  is not  $c$ -guaranteed, and hence  $[\mu_1, \dots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \dots, \mu_d]$  is not  $c$ -guaranteed, for any  $j \in [d]$ .

Finally, it is evident that all  $j \in [d]$  are pinch points for  $[\mu_1, \dots, \mu_d]$ . (It is interesting to note that  $[\mu_1, \dots, \mu_d]$  is the lexicographically first element of  $\mathcal{O}(c, d)$ .)  $\square$

## 6 Upper bound on the size of the obstruction set

It was shown in [4] that  $|\mathcal{O}(c, 2)| \leq 2c^2$ . We give an asymptotic upper bound for  $|\mathcal{O}(c, d)|$  for every fixed  $d \geq 3$ .

**Theorem 6.1.** *For all  $d \geq 3$ ,*

$$|\mathcal{O}(c, d)| = O\left(c^{(17 \cdot 3^{d-3} - 1)/2}\right).$$

*Proof.* Fix  $d \geq 3$ . We give an asymptotic upper bound on the number of monotone grids in  $\mathcal{O}(c, d)$ . The size of  $\mathcal{O}(c, d)$  is at most  $d!$  times this bound, and so it is asymptotically equivalent. By Lemma 5.2, every grid  $R \in \mathcal{O}(c, d)$  has a set  $P$  of pinch points. For each set  $P \subseteq [d]$  such that  $d \in P$ , let  $\#_c(P)$  be the number of monotone grids in  $\mathcal{O}(c, d)$  having pinch point set  $P$ . There are  $2^{d-1}$  many such  $P$ , so an asymptotic bound on  $\max\{\#_c(P) \mid P \subseteq [d] \wedge d \in P\}$  gives the same asymptotic bound on  $|\mathcal{O}(c, d)|$ .

Fix a set  $P \subseteq [d]$  such that  $d \in P$ , and let  $P = \{\ell_1 < \ell_2 < \dots < \ell_s = d\}$ , where  $s = |P|$  and  $\ell_1, \dots, \ell_s$  are the elements of  $P$  in increasing order. For convenience, set  $\ell_0 := 0$ . Lemma 5.2 says that for any monotone grid  $R = [a_1, \dots, a_d] \in \mathcal{O}(c, d)$  having pinch point set  $P$ , for any  $b \in [s]$ , and for any  $k$

such that  $\ell_{b-1} < k \leq \ell_b$ , we have  $a_k = O(c^{e(b)})$ , where

$$e(b) := 3^{\ell_{b-1}} \cdot 2^{\ell_b - \ell_{b-1} - 1}.$$

To bound  $\#_c(P)$ , we first note that for any choice of  $1 \leq a_1 \leq \dots \leq a_{d-1}$ , there can be at most one value of  $a_d$  such that  $[a_1, \dots, a_d] \in \mathcal{O}(c, d)$ , because any two  $d$ -dimensional grids that share the first  $d-1$  dimensions are comparable in the dominance order  $\preceq$ . Thus  $\#_c(P)$  is bounded by the number of possible combinations of values of  $a_1, \dots, a_{d-1}$ . From the bound on each  $a_k$  above, we therefore have

$$\begin{aligned} \#_c(P) &\leq \left( \prod_{b=1}^{s-1} \prod_{k=\ell_{b-1}+1}^{\ell_b} O(c^{e(b)}) \right) \cdot \prod_{k=\ell_{s-1}+1}^{d-1} O(c^{e(s)}) \\ &= O\left( \prod_{b=1}^{s-1} (c^{e(b)})^{\ell_b - \ell_{b-1}} \right) \cdot O\left( (c^{e(s)})^{d-1 - \ell_{s-1}} \right) \\ &= O(c^{h_1 + h_2}) \end{aligned}$$

where  $h_2 = e(s)(d-1 - \ell_{s-1})$  and

$$\begin{aligned} h_1 &= \sum_{b=1}^{s-1} e(b)(\ell_b - \ell_{b-1}) \\ &= \sum_{b=1}^{s-1} 3^{\ell_{b-1}} \cdot 2^{\ell_b - \ell_{b-1} - 1} \cdot (\ell_b - \ell_{b-1}) \\ &\leq \sum_{b=1}^{s-1} 3^{\ell_{b-1}} \cdot \frac{3^{\ell_b - \ell_{b-1}} - 1}{2} \\ &= \frac{1}{2} \sum_{b=1}^{s-1} (3^{\ell_b} - 3^{\ell_{b-1}}) \\ &= \frac{3^m - 1}{2}, \end{aligned}$$

where  $m = \ell_{s-1}$ . We also have

$$\begin{aligned} h_2 &= 3^{\ell_{s-1}} \cdot 2^{d - \ell_{s-1} - 1} \cdot (d - 1 - \ell_{s-1}) \\ &= 3^m \cdot 2^{d-m-1} \cdot (d - m - 1), \end{aligned}$$

whence

$$h_1 + h_2 = \frac{3^m - 1}{2} + 3^m \cdot 2^{d-m-1} \cdot (d - m - 1).$$

So our bound on the exponent of  $c$  only depends on the value of  $m$ , which satisfies  $0 \leq m < d$ . It is more convenient to express  $h_1 + h_2$  in terms of  $n := d - m$ , where  $n \in [d]$ :

$$h_1 + h_2 = \frac{3^{d-n} - 1}{2} + 3^{d-n} \cdot 2^{n-1} \cdot (n - 1)$$

$$= \frac{3^d}{2} \cdot \frac{1 + 2^n(n-1)}{3^n} - \frac{1}{2}.$$

It is easy to check that  $(1 + 2^n(n-1))/3^n$  is greatest (and thus  $h_1 + h_2$  is greatest) when  $n = 3$ . It follows that

$$\begin{aligned} h_1 + h_2 &\leq \frac{3^d}{2} \cdot \frac{1 + 2^3(3-1)}{3^3} - \frac{1}{2} \\ &= \frac{17 \cdot 3^{d-3} - 1}{2}, \end{aligned}$$

which proves the theorem.  $\square$

The first few values  $(17 \cdot 3^{d-3} - 1)/2$  are given in the Figure 1.

$d$	$(17 \cdot 3^{d-3} - 1)/2$
3	8
4	25
5	76
6	229

Figure 1: Table of upper bounds on  $e$  so that  $|\mathcal{O}(c, d)| = O(c^e)$  for small  $d$ .

## 7 Three Dimensions and Two Colors

The following graph (Figure 2, generated using the Jmol module in SAGE) and table (Figure 3) display upper bounds for the smallest  $a_3$  so that  $[a_1, a_2, a_3]$  is 2-guaranteed. All three graphical axes run from 3 to 130; the table includes only  $3 \leq a_1 \leq 12$  and  $3 \leq a_2 \leq 12$ . We believe these values to be very close to the truth; indeed, we have matching lower bounds in many cases, and lower bounds that differ from the upper bounds by at most 2 in many more cases.

A few different methods were applied to obtain these bounds. First, the values  $\Delta_j$ , as in Section 3, were computed, and the least  $a_3$  so that  $\Delta_3 > 0$  was recorded. In fact, this idea was improved slightly by applying the observation that, if some grid is  $(2, t)$ -guaranteed, then it is  $(2, \lceil t \rceil)$ -guaranteed. In some cases, this increases the value of  $\Delta_j$ . Second, we used the simple observations that  $c$ -colorability is independent of the order of the  $a_i$ , and that  $R \preceq R'$  when  $R$  is  $c$ -guaranteed implies that  $R'$  is  $c$ -guaranteed. Third, we applied the following lemma.

**Lemma 7.1.** *If the grid  $R = [a_1, \dots, a_d]$  is  $(c, t)$ -guaranteed, then  $R \times [\lceil cM/t \rceil + 1]$  is  $c$ -guaranteed, where  $M = \prod_{j=1}^d \binom{a_j}{2}$*

*Proof.* Note that  $K = \lceil cM/t \rceil + 1 > cM/t$  and is integral. If we think of  $R \times [K]$  as  $K$  copies of  $R$ , then any  $c$ -coloring of  $R \times [K]$  restricts to  $K$   $c$ -colorings of  $R$ . Since  $R$  is  $(c, t)$ -guaranteed, each of these  $c$ -colorings gives rise to

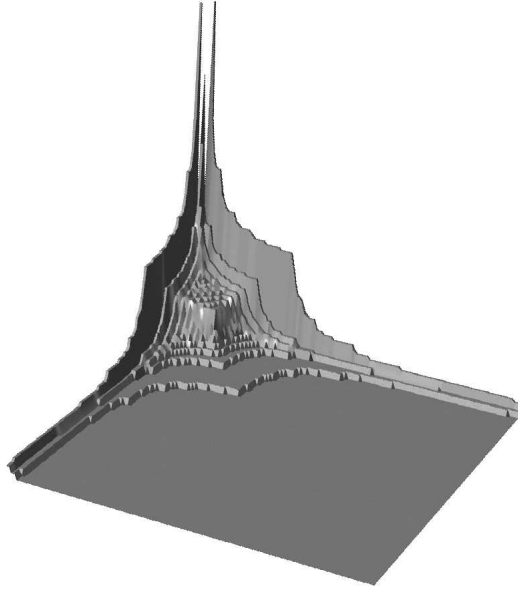


Figure 2: Graph of upper bounds on  $a_3$  so that  $[a_1, a_2, a_3]$  is 2-guaranteed.

	3	4	5	6	7	8	9	10	11	12
3					127	85	73	68	67	67
4					127	85	73	68	67	67
5			101	76	53	47	46	46	40	37
6			76	76	53	47	46	46	40	37
7	127	127	53	53	53	46	40	37	34	33
8	85	85	47	47	46	45	40	37	34	33
9	73	73	46	46	40	40	37	34	31	30
10	68	68	46	46	37	37	34	33	31	30
11	67	67	40	40	34	34	31	31	30	28
12	67	67	37	37	33	33	30	30	28	28

Figure 3: Table of bounds on  $a_3$  so that  $[a_1, a_2, a_3]$  is 2-guaranteed.

$t$  monochromatic boxes. Hence, in  $K$  colorings, there are at least  $t(\lfloor cM/t \rfloor + 1) > cM$  monochromatic boxes. Since there are only  $M$  total boxes in each copy of  $R$ , and any monochromatic box can only be colored in  $c$  different ways, there must be two identical boxes (in two different copies of  $R$ ) which are monochromatic and have the same color. This is precisely a monochromatic  $(d+1)$ -dimensional box in  $R \times [K]$ .  $\square$

Therefore, in order to obtain upper bounds on  $[a_3]$  in the above table, we need to know the greatest  $t$  for which  $[a_1] \times [a_2]$  is  $(2, t)$ -guaranteed. To that end, we define the following matrix:

**Definition 7.2.** Let  $M_r$  be the  $2^r \times 2^r$  integer matrix whose rows and columns are indexed by all maps  $f_j : [r] \rightarrow [2]$ ,  $0 \leq j < 2^r$ . The  $(i, j)$ -entry of  $M_r$  is defined to be

$$\binom{|f_i^{-1}(1) \cap f_j^{-1}(1)|}{2} + \binom{|f_i^{-1}(2) \cap f_j^{-1}(2)|}{2}.$$

Then define the quadratic form  $Q_r : \mathbb{R}^{2^r} \rightarrow \mathbb{R}$  by  $Q_r(\mathbf{v}) = \mathbf{v}^* M_r \mathbf{v}$ . Let  $\delta_r = (M_r(1, 1), \dots, M_r(2^r, 2^r))$ , the diagonal of  $M_r$ .

**Proposition 7.3.** *Let  $t$  be the least value of  $Q_r(\mathbf{v}) - \mathbf{v} \cdot \delta_r$  over all nonnegative integer vectors  $\mathbf{v} \in \mathbb{Z}^{2^r}$  with  $\mathbf{v} \cdot \mathbf{1} = s$ . Then  $[r] \times [s]$  is  $(c, t)$ -guaranteed, and  $t$  is the minimum value so that this is the case.*

*Proof.* Given a vector  $\mathbf{v} = (v_1, \dots, v_r)$  satisfying the hypotheses, consider the  $r \times s$  matrix  $A$  with  $v_j$  columns of type  $f_j$  for each  $j \in [r]$ . (We may identify  $f_j$  with a column vector in  $[2]^r$  in the natural way.) It is easy to see that  $Q_r(\mathbf{v}) - \delta_r$  exactly counts twice the number of monochromatic rectangles in  $A$ , thought of as a 2-coloring of the grid  $[r] \times [s]$ .  $\square$

We applied standard quadratic integer programming tools (XPress-MP) to minimize the appropriate programs. Fortunately, for the cases considered, the matrix  $M_r$  was positive semidefinite, meaning that the solver could use polynomial time convex programming techniques during the interior point search. We conjecture that this is always the case.

**Conjecture 7.4.**  *$M_r$  is positive semidefinite for  $r \geq 3$ .*

In particular, for  $r = 3$ , the eigenvalues of  $M_r$  are 0, 1, and 4, with multiplicities 2, 4, and 2, respectively. For  $4 \leq r \leq 9$ , the eigenvalues are 0,  $2^{r-2}$ ,  $2^{r-3}(r-2)$ ,  $2^{r-2}(r-1)$ , and  $2^{r-4}(r^2-r+2)$ , with multiplicities  $2^r - r(r+1)/2$ ,  $r(r-1)/2 - 1$ ,  $r-1$ , 1, and 1, respectively. We conjecture that this description of the spectrum is valid for all  $r \geq 4$ .

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