Math 784
Homework 3

Due Friday, March 7.

Problems:

1. - Let $K = \mathbb{Q}(\alpha)$, where α is a root of the polynomial $f_\alpha(x) = x^3 - 2x + 3$. Give the prime factorization in O_K of the ideals $211O_K$, $2O_K$, $3O_K$, and $7O_K$. (Note: It was shown in Homework 2, problem 4 that $O_K = \mathbb{Z}[\alpha]$.)
   - Let $K = \mathbb{Q}(\sqrt{-5})$. In Homework 1, problem 4, it was shown that $O_K = \mathbb{Z}[\sqrt{-5}]$ is not a UFD since, for example,

     $$2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

     gives distinct factorizations of $6 \in O_K$. Factor the ideals $2O_K$, $3O_K$, $(1 + \sqrt{-5})O_K$, and $(1 - \sqrt{-5})O_K$ in O_K.
   - Let $\zeta_7 = e^{\frac{2\pi i}{7}}$, let $K = \mathbb{Q}(\zeta_7)$, and let $p \in \mathbb{Z}$ be prime. Describe the prime factorization in O_K of the ideals pO_K. Your answer should depend on $p \mod 7$.

2. - Let d be a positive square-free integer with $d \equiv 1, 2 \pmod 4$, let $K = \mathbb{Q}(\sqrt{-d})$, and let $p \in \mathbb{Z}$ be an odd prime with $p \nmid d$. Show that there are $x, y \in \mathbb{Z}$ with $p = x^2 + dy^2$ if and only if p splits completely in O_K into principal prime ideals.
   - Let $p \in \mathbb{Z}$ be prime. Show that there are $x, y \in \mathbb{Z}$ with $p = x^2 + y^2$ if and only if $p \equiv 1 \pmod 4$ or $p = 2$.
   - Let $p \in \mathbb{Z}$ be prime. Show that there are $x, y \in \mathbb{Z}$ with $p = x^2 + 2y^2$ if and only if $p \equiv 1, 3 \pmod 8$ or $p = 2$.

3. Marcus §2, #2.

4. Marcus §3, #10.

5. Marcus §3, #30 a, b, c.

6. Let p be prime, let $|a| > 1$ be a square-free integer with $p \nmid a$, let $\theta = a^{1/p}$, and let $K = \mathbb{Q}(\theta)$. Show that $O_K = \mathbb{Z}[\theta]$ if and only if $a^{p-1} \not\equiv 1 \pmod{p^2}$.