Math 544, Exam 3 Information

Exam 3 will be based on:

- Sections 4.1 - 4.5, 4.7, and 5.2, 5.3.
- The corresponding assigned homework problems
(see http://www.math.sc.edu/∼boylan/SCCourses/math5443/544.html).
At minimum, you need to understand how to do the homework problems.
- Lecture notes: 10/29 - 11/26.
- Quizzes: 7 - 10.

Topic List (not necessarily comprehensive):

You will need to know how to define vocabulary words/phrases defined in class.

§**4.1**: **The eigenvalue problem for** 2×2 **matrices:** Definition and computation of eigenvalues and eigenvectors for 2×2 matrices.

§**4.2**: **Determinants and the eigenvalue problem:** Definition and computation of determinants of matrices $A \in \text{Mat}_{n\times n}(\mathbb{R})$. Computation of determinants by expansion across rows or down columns using **minors** and **cofactors**. What is the **minor** and **cofactor** associated to a matrix entry (a_{ij}) of A? Properties of determinants, for example:

- $\det(AB) = \det(A)\det(B)$.
- $A \in \text{Mat}_{n\times n}(\mathbb{R})$ is singular $\iff$ $\det(A) = 0$.
- If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

What is the determinant of a **triangular** matrix?

§**4.3**: **Elementary operations and determinants:** Important properties: $\det(A) = \det(A^{\mathrm{T}})$, $\det(cA) = c^n\det(A)$ for $c \neq 0$ in $\mathbb{R}$. Effects of elementary row **and** column operations on the computation of a determinant:

- Interchanging two rows or two columns changes the sign of the determinant.
- If the row operation
$$R_i \mapsto \frac{1}{k}R_i, \quad k \neq 0$$
transforms matrix A into matrix B, then $\det(A) = k\det(B)$. (In effect, you are "factoring" k out of the ith row of A.) Similarly, if the column operation
$$C_i \mapsto \frac{1}{k}C_i, \quad k \neq 0$$
transforms matrix A into matrix B, then $\det(A) = k\det(B)$.

- A row operation of the form

$$R_i \mapsto R_i + kR_j, \;\; k \neq 0, \;\; i \neq j$$

does nothing to the determinant. Similarly, a column operation of the form

$$C_i \mapsto C_i + kC_j, \;\; k \neq 0, \;\; i \neq j$$

does nothing to the determinant.

§**4.4**: **Eigenvalues and the characteristic polynomial:** The definition and computation of the **eigenvalues** of a matrix $A \in \text{Mat}_{n\times n}(\mathbb{R})$. i.e., computation of the **characteristic polynomial** $p(t) = \det(A - tI_n)$; the **algebraic multiplicity** of an eigenvalue λ is the number of times the factor $(t - \lambda)$ occurs in the characteristic polynomial $p(t)$.

- If λ is an eigenvalue of A and $k \geq 0$ is an integer, then λ^k is an eigenvalue of A^k.
- If λ is an eigenvalue of A and $\alpha \in \mathbb{R}$, then $\lambda + \alpha$ is an eigenvalue of $A + \alpha I_n$.
- If A is invertible and λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.
- If λ is an eigenvalue of A, then it is also an eigenvalue of A^{T}.
- A matrix A has 0 as one of its eigenvalues if and only if it is singular.

What are the eigenvalues of a triangular matrix?

§**4.5**: **Eigenspaces and eigenvectors:** The definition and computation of the eigenvectors of a matrix $A \in \text{Mat}_{n\times n}(\mathbb{R})$. If λ is an eigenvalue of $A \in \text{Mat}_{n\times n}(\mathbb{R})$, then the **eigenspace** associated to λ is $E_\lambda = \text{Null}(A - \lambda I)$ and the **geometric multiplicity** of λ is the dimension of E_λ (i.e., the nullity of $A - \lambda I$). The relationship between algebraic and geometric multiplicities is

$$1 \leq \text{geometric mult.}(\lambda) \leq \text{algebraic mult.}(\lambda).$$

Definition of a **defective** matrix: a matrix A is defective if A has at least one eigenvalue whose geometric mult. is **strictly less than** its algebraic mult. i.e., there is an eigenvalue λ with

$$\text{geom. mult.}(\lambda) < \text{alg. mult.}(\lambda).$$

Important fact: Eigenvectors associated to distinct eigenvalues are linearly independent. As a consequence, if $A \in \text{Mat}_{n\times n}(\mathbb{R})$ is not defective, then A has n linearly independent eigenvectors, and these eigenvectors form a basis for $\mathbb{R}^n$. In particular,, if A has n distinct eigenvalues, then A is not defective.

§**4.7**: **Similarity transformations and diagonalization:** Matrices A and $B \in \text{Mat}_{n\times n}(\mathbb{R})$ are **similar** if there is an invertible matrix S for which

$$B = S^{-1}AS.$$

A matrix A is **diagonalizable** if it is similar to a diagonal matrix D. If A and B are similar, they have the same:

- characteristic polynomial: $p_A(t) = p_B(t)$. **However, the converse is not true:** If $p_A(t) = p_B(t)$, then it is not always true that A and B are similar.
- eigenvalues and algebraic multiplicities (**but the corresponding eigenvectors are typically different!** If $B = S^{-1}AS$ (so A and B are similar) and if $\vec{\mathbf{x}}$ is an eigenvector of B associated to λ (so $B\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$), then $S\vec{\mathbf{x}}$ is an eigenvector of A associated to λ (so $A(S\vec{\mathbf{x}}) = \lambda(S\vec{\mathbf{x}})$)).

Criterion for diagonalizability: The diagonalizability of A is equivalent to

- A has n linearly independent eigenvectors (the maximum possible).
- A is not defective (i.e., the geometric and algebraic multiplicities agree for all eigenvalues of A). So a matrix A is either defective or diagonalizable.

If A is diagonalizable, then there is an invertible matrix S and a diagonal matrix D for which

$$D = S^{-1}AS.$$

How do you find the matrices S and D?

- Compute the eigenvalues of A and their algebraic multiplicities. Suppose that the **distinct** eigenvalues of A are $\lambda_1, \ldots, \lambda_k$.
- Compute bases $\mathcal{B}_1, \ldots, \mathcal{B}_k$ for the eigenspaces $E_{\lambda_1}, \ldots, E_{\lambda_k}$. The dimension of E_{λ_i} is the geometric multiplicity of λ_i. If for all i,

  $$\text{alg. mult.}(\lambda_i) = \text{geom. mult.}(\lambda_i),$$

  then A is diagonalizable.
- If A is diagonalizable, form the set $\mathcal{B} = \{\vec{\mathbf{w}}_1, \ldots, \vec{\mathbf{w}}_n\}$ consisting of all the basis vectors for the eigenspaces of A. Then the invertible matrix S which diagonalizes A is

  $$S = (\vec{\mathbf{w_1}} \mid \vec{\mathbf{w_2}} \mid \cdots \mid \vec{\mathbf{w_n}}).$$

  So we have

  $$D = S^{-1}AS,$$

  where D is a diagonal matrix with diagonal entry $(D)_{ii} = \lambda_i$ and λ_i is the eigenvalue of A associated to the eigenvector $\vec{\mathbf{w_i}}$: $A\vec{\mathbf{w_i}} = \lambda_i\vec{\mathbf{w_i}}$.

If A is diagonalizable, and $k \geq 0$ is an integer, how can you compute A^k? Here's how: A diagonalizable implies that for some invertible matrix S, $D = S^{-1}AS$ is diagonal. We then have $D^k = (S^{-1}AS)^k = S^{-1}D^kS$. Moving the S's to the left side, we obtain $SD^kS^{-1} = A^k$. So if you know S and S^{-1} (it is easy to compute D^k if D is diagonal), you can compute A^k.

Orthogonal matrices: Their definition and basic properties:

- $Q \in \text{Mat}_{n\times n}(\mathbb{R})$ is orthogonal if and only if its rows and columns form orthonormal bases for $\mathbb{R}^n$.
- If you rearrange the rows or columns of an orthogonal matrix, the resulting matrix is still orthogonal.
- If Q is orthogonal, then:
  - $\forall \vec{\mathbf{x}} \in \mathbb{R}^n$, $\|Q\vec{\mathbf{x}}\| = \|\vec{\mathbf{x}}\|$. (Multiplication by Q preserves length)
  - $\forall \vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$, $Q\vec{\mathbf{x}} \cdot Q\vec{\mathbf{y}} = \vec{\mathbf{x}} \cdot \vec{\mathbf{y}}$. (Multiplication by Q preserves the angle between vectors.)
  - $\det(Q) = \pm 1$.
- $A \in \text{Mat}_{n\times n}(\mathbb{R})$ is symmetric if and only if it is **orthogonally diagonalizable**. i.e., $\exists Q$, orthogonal, such that $Q^{-1}AQ = D$ is diagonal.

§**5.2**: **Vector spaces:** The definition of vector space (a set V and a scalar field F together with an addition operation on V and a scalar multiplication operation); in particular, the ten vector space axioms: 2 closure axioms, 4 axioms for vector addition, 4 axioms for scalar multiplication. Examples of vector spaces: $\text{Mat}_{m\times n}(\mathbb{R})$, P_n. Check whether a set V together with an addition and scalar multiplication is or is not a vector space.

§**5.3**: **Subspaces:** Determination of whether or not certain subsets of $\text{Mat}_{m\times n}(\mathbb{R})$ or P_n are subspaces.