Math 544, Exam 2 Information

Exam 2 will be based on:

- Sections 3.2 - 3.7.
- The corresponding assigned homework problems
  (see http://www.math.sc.edu/∼boylan/SCCourses/math5443/544.html).
  At minimum, you need to understand how to do the homework problems.
- Lecture notes: 9/26 - 10/24.

Topic List (not necessarily comprehensive):

You will need to know how to define vocabulary words/phrases defined in class.

§**3.2**: **Vector space axioms:**

- closure properties (2)
- addition properties (4)
- multiplication properties (4)

Subspaces: How do you determine whether a subset W of a vector space V is a subspace?

1. To show that W is a subspace, you need to verify the three subspace axioms.
2. To show that W is not a subspace, it suffices to provide a simple numerical example in which one of the axioms is violated.

Note: If a subset W of $\mathbb{R}^n$ is defined by simultaneous linear homogeneous equations (i.e., a homogeneous system), then it is a subspace.

§**3.3**: **Examples of subspaces:** What is the **Span** of a set of vectors?

Let $A \in \text{Mat}_{m\times n}(\mathbb{R})$. Be able to define the important subspaces associated to A. These are:

- The **null space** of A (a subspace of $\mathbb{R}^n$).
- The **range** of A (a subspace of $\mathbb{R}^m$).
- The **column space** of A (a subspace of $\mathbb{R}^m$ which coincides with the range of A).
- The **row space** of A (a subspace of $\mathbb{R}^n$).

Problem: Given a matrix A, give a basis (algebraic description) for each of these subspaces.

Fact: If A and B are row equivalent $m \times n$ matrices, then they have the same row space.

§**3.4**: **Bases for subspaces:** What is a **basis** for a subspace W of $\mathbb{R}^n$?

Fact: If B is a basis for W, then every vector $w \in W$ has a unique representation as a linear combination of vectors from B.

Given a matrix $A \in \text{Mat}_{m\times n}(\mathbb{R})$, compute bases for Null$(A)$, Range$(A)$, Col$(A)$, and Row$(A)$. To do this, you first compute the reduced echelon form of A. Call it B.

- **Row**(A)**:** The nonzero rows of B form a basis for Row(A).
- **Col**(A)**:** The columns of B with the leading 1's correspond to the columns of A which form a basis.
- **Range**(A)**:** Since the range and column space of A agree, you compute Range(A) just as you would Col(A).
- **Null**(A)**:** Null(A) is the set of solutions to the homogenous system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Therefore, begin by solving $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Convert your solution to vector form. i.e., write your solution as Null(A) = Span(S). Verify that the vectors in S are linearly independent (which is usually easy to do and requires little or no justification).

§**3.5**: **Dimension:** Let W be a subspace of $\mathbb{R}^n$.

Facts:

- W has many bases, but the number of vectors in every basis is the same. This number is called the **dimension** of W.
- If $A \in \text{Mat}_{m\times n}(\mathbb{R})$, then rank$(A)$ + nullity$(A) = n$. (what is rank(A)? nullity(A)?).
- rank(A) = dim(Col(A)) = dim(Row(A)).
- $A \in \text{Mat}_{n\times n}(\mathbb{R})$ is non-singular if and only if rank$(A) = n$ and nullity$(A) = 0$.

Be able to compute rank(A)**, nullity**(A).

Know Theorem 3.9, page 207.

§**3.6**: **Orthogonal bases for subspaces:** Define the terms **orthogonal** and **orthonormal**. How do you convert an orthogonal set to an orthonormal one? If B is a basis for a subspace W of $\mathbb{R}^n$ consisting of orthogonal vectors, how do you express a vector $\vec{\mathbf{w}} \in W$ as a linear combination of basis vectors? Must a basis of orthogonal vectors for W **always** exist? Be able to use the Gram-Schmidt process to convert a basis B to an orthogonal basis, B'.

§**3.7**: **Linear transformations from** $\mathbb{R}^n$ **to** $\mathbb{R}^m$**:** Define what it means for a function $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ to be a linear transformation. Given a function $T : \mathbb{R}^n \mapsto \mathbb{R}^m$, how does one check whether T is a linear transformation?

1. To show that T is a linear transformation, you need to verify the two linear transformation axioms.
2. To show that T is not a linear transformation, it suffices to provide a simple numerical example in which one of the axioms is violated.

Fact: If $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation, then for every $\vec{\mathbf{x}} \in \mathbb{R}^n$, $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$, where

$$A = [T(\vec{\mathbf{e_1}}) \mid \cdots \mid T(\vec{\mathbf{e_n}})]$$

(i.e., it suffices to know what T does to the standard basis for $\mathbb{R}^n$).

What are the **Null space** and **Range** of T?