Math 784, Algebraic Number Theory
Spring 2010

- **Instructor:** Matthew Boylan
- **Course Description:**

This course is an introduction to algebraic number theory. Algebraic number theory is one of the foundations of modern number theory. It is primarily the study of **number fields**, which are finite algebraic extensions of the rationals. Of particular interest are primes, (prime ideals, really), rings of integers (the analogue of $\mathbb{Z}$ in number fields), and units in number fields. If K is a number field, one can often deduce a terrific amount of information about K by studying its extensions of finite degree.

For example, we now know by work of Wiles et al that if $n \geq 3$, then the Fermat equation

$$x^n + y^n = z^n, \quad xyz \neq 0$$

has no solution in integers. Some cases of this result follow quickly from elementary algebraic number theory, a fact proved by Kummer in 1850. He showed that if n is divisible by an odd prime p which does not divide the **class number** (we will study the class number in this course) of the field $\mathbb{Q}(e^{\frac{2\pi i}{p}})$, then the Fermat equation with exponent n has no solution in integers. Such primes p are called regular primes, and it is not known whether there are infinitely many. Nevertheless, heuristics predict that about 65% of the primes are regular. As an example, 7 is regular since $\mathbb{Q}(e^{\frac{2\pi i}{7}})$ has class number 1 and $7 \nmid 1$, so the Fermat equation has no solution in integers for any exponent divisible by 7.

Here is a tentative list of topics to be covered:

• Basics on number fields: primitive element theorem, discriminants, norms, traces, rings of integers, bases.
• Quadratic fields ($\mathbb{Q}(\sqrt{d})$, d square-free) and cyclotomic fields ($\mathbb{Q}(e^{\frac{2\pi i}{n}})$, $n \geq 1$).
• Unique factorization of **ideals** in number fields. The ideal class group and class number (the order of the ideal class group). The **elements** in a number field factor uniquely if and only if the number field has class number 1. Geometry of numbers and finiteness of class group.
• Dirichlet's Unit Theorem.
• Splitting of primes in number fields: inertia and decomposition groups; Frobenius endomorphism. (This is where Galois theory is needed.)
• (time permitting) Statements of the basic theorems of class field theory: Kronecker-Weber Theorem (a Galois number field with abelian Galois group is a subfield of some cyclotomic field), Chebotarev Density Theorem, Artin reciprocity (quadratic reciprocity is a very, very special case), Hilbert class fields.

- **Text:**

  Marcus, Daniel A. Number fields. Universitext. Springer-Verlag, New York-Heidelberg, 1977. viii+279 pp.

- **Other References:**

  (1) Cox, David A. Primes of the form $x^2 + ny^2$. Fermat, class field theory and complex multiplication. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989. xiv+351 pp.
  (2) Fröhlich, A.; Taylor, M. J. Algebraic number theory. Cambridge Studies in Advanced Mathematics, 27. Cambridge University Press, Cambridge, 1993. xiv+355 pp.
  (3) Janusz, Gerald J. Algebraic number fields. Second edition. Graduate Studies in Mathematics, 7. American Mathematical Society, Providence, RI, 1996. x+276 pp.
  (4) Lang, Serge. Algebraic number theory. Second edition. Graduate Texts in Mathematics, 110. Springer-Verlag, New York, 1994. xiv+357 pp.
  (5) Neukirch, Jürgen. Algebraic number theory. Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by G. Harder. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 322. Springer-Verlag, Berlin, 1999. xviii+571 pp.
  (6) Swinnerton-Dyer, H. P. F. A brief guide to algebraic number theory. London Mathematical Society Student Texts, 50. Cambridge University Press, Cambridge, 2001. x+146 pp.

- **Homework:** Homework will be assigned weekly or once every two weeks.
- **Grading:** Grading will be based on homework. Mid-term and final exams will be administered in formats to be determined.