Homework 6 Solutions.

§6.1 #1 c. Show that $\sqrt{3}+\sqrt{5}$ is algebraic over $\mathbb{Q}$.

Proof. Compute as follows:

$$\begin{aligned} a = \sqrt{3}+\sqrt{5} &\iff a^2 = 3+2\sqrt{15}+5 = 8+2\sqrt{15} \iff (a^2-8)^2 = 4\cdot 15 = 60 \\ &\iff a^4 - 16a^2 + 4 = 0. \end{aligned}$$

Hence, for $f(x) = x^4 - 16x^2 + 4 \in \mathbb{Q}[x]$, we have $f(a) = 0$, so a is algebraic over $\mathbb{Q}$. □

§6.1 #1 e. Show that $\frac{-1+i\sqrt{3}}{2}$ is algebraic over $\mathbb{Q}$.

Proof. Compute as follows:

$$\begin{aligned} a = \frac{-1+i\sqrt{3}}{2} &\iff 2a+1 = i\sqrt{3} \iff (2a+1)^2 = -3 \iff 4a^2+4a+4 = 0 \\ &\iff a^2+a+1 = 0. \end{aligned}$$

Hence, for $f(x) = x^2+x+1 \in \mathbb{Q}[x]$, we have $f(x) = 0$, so a is algebraic over $\mathbb{Q}$. □

§6.1 #1 f. Show that $\sqrt[3]{2}+\sqrt{2}$ is algebraic over $\mathbb{Q}$.

Proof. Compute as follows:

$$\begin{aligned} & a = \sqrt[3]{2}+\sqrt{2} \iff a-\sqrt{2} = \sqrt[3]{2} \\ &\iff (a-\sqrt{2})^3 = a^3 - 3a^2\sqrt{2} + 3a\cdot 2 - \sqrt{2}^3 = a^3 + 6a - (3a^2+2)\sqrt{2} = 2 \\ &\iff a^3 + 6a - 2 = (3a^2+2)\sqrt{2} \\ &\iff a^6 + 12a^4 - 4a^3 + 36a^2 - 24a + 4 = 18a^4 + 24a^2 + 8 \\ &\iff a^6 - 6a^4 - 4a^3 + 12a^2 - 24a - 4 = 0. \end{aligned}$$

Hence, for $f(x) = x^6 - 4x^3 + 12x^2 - 24x - 4 \in \mathbb{Q}[x]$, we have $f(a) = 0$, so a is algebraic over $\mathbb{Q}$. More efficient ways to do this problem exist, but require theoretical tools not yet developed. □

§6.1 #2 Let $F \subseteq K$ be fields, and let $u \neq 0$ in F be algebraic over K. Show that u^{-1} is algebraic over K.

Proof. Since u is algebraic over K, there exists a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots a_1 x + a_0 \in K[x]$ with $f(u) = a_n u^n + \cdots a_1 u + a_0 = 0$. Multiplying by u^{-n} yields $a_n + a_{n-1}u + \cdots a_1 u^{n-1} + a_0 u^n = 0$. Hence, we see that u^{-1} satisfies the **reciprocal polynomial** of f given by $g(x) = a_0 x^n + a_1 x^{n-1} + \cdots a_{n-1}x + a_n$. It follows that u^{-1} is algebraic. □

§6.1 #3 Suppose that u is algebraic over the field K, and that $a \in K$. Show that $u + a$ is algebraic over K, find the minimal polynomial of $u + a$, and show that $u + a$ and u have the same degree over K.

Proof. Let $f_u(x) \in K[x]$ be the minimal polynomial for u over K. Then $f_u(x)$ is monic and irreducible in $K[x]$. Therefore, for all $a \in K$, the polynomial $g_{u,a}(x) := f_u(x - a) \in K[x]$ must also be monic and irreducible in $K[x]$. Moreover, these polynomials have the same degree and $g_{u,a}(u + a) = f_u(u) = 0$. It follows that $g_{u,a}(x) = f_{u+a}(x)$, the minimal polynomial of $u + a$ over K. Since $f_{u+a}(x)$ and $f_u(x)$ have the same degree, the degrees of u and $u + a$ over K are the same. □

§6.1, #4 Show that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$.

Proof. Suppose that $\sqrt{3} \in \mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Then there are c, $d \in \mathbb{Q}$ with $\sqrt{3} = c + d\sqrt{2}$. Squaring both sides yields $3 = c^2 + 2cd\sqrt{2} + 2d$. We conclude that

$$\sqrt{2} = \frac{3 - c^2 - 2d}{2cd} \in \mathbb{Q},$$

a contradiction since 2 is square-free. □

§6.1 #5 (a) Show that $f(x) = x^3 + 3x + 3$ is irreducible over $\mathbb{Q}$.

Proof. Since $f(x)$ is 3-Eisenstein, it irreducible over $\mathbb{Q}$. □

(b) Let u be a root of $f(x)$. Express u^{-1} and $(1 + u)^{-1}$ in the form $a + bu + cu^2$ for some a, b, $c \in \mathbb{Q}$.

Solution: Since $f(u) = u^3 + 3u + 3 = 0$, multiplying by u^{-1} gives $u^2 + 3 + 3u^{-1} = 0$. It follows that $u^{-1} = -\frac{1}{3} \cdot u^2 - 1$.

Note also that $u^3 + 3u + 3 = u^3 + 3(u + 1) = 0$ implies that $u^3 + 1 + 3(u + 1) = 1$. Factoring the left-hand side gives

$$u^3 + 1 + 3(u + 1) = (u + 1)(u^2 - u + 1) + 3(u + 1) = (u + 1)(u^2 - u + 4) = 1.$$

It follows that $(u + 1)^{-1} = u^2 - u + 4$.

§6.1 #9 Let $F \supseteq K$ be fields. If $u \in F$ is transcendental over K, then show that every element of $K(u)\backslash K$ is transcendental over K.

Proof. Suppose that $v \in K(u)\backslash K$. Since u is transcendental, there exists $f(x), g(x) \in K[x]$ with $g(u) \neq 0$ and $v = \frac{f(u)}{g(u)}$. Since $v \notin K$, the rational function $\frac{f(x)}{g(x)}$ is non-constant.

Now, we argue by contradiction. Suppose that v is algebraic over K. Then there is a non-constant polynomial $h(x) \in K[x]$ with degree $n \geq 1$ and with $h(v) = 0$. Recalling that $v = \frac{f(u)}{g(u)}$, we obtain $h\left(\frac{f(u)}{g(u)}\right) = 0$. Note that $g(x)^n \cdot h\left(\frac{f(x)}{g(u)}\right) \in K[x]$ is a polynomial satisfied by u. (Multiplying by $g(x)^n$ clears denominators.) Hence, u must be algebraic over K, contradicting the hypothesis. □

§6.1 #10 Let u and $r \in \mathbb{R}$ be positive with $u \neq 1$. A theorem of Gelfand and Schneider asserts that if r is irrational and u and r are algebraic over $\mathbb{Q}$, then u^r is transcendental over $\mathbb{Q}$. Use this theorem to show:

(a) $\sqrt[3]{7}^{\sqrt{5}}$ is transcendental.

Proof. We observe that $\sqrt[3]{7}$ satisfies $x^3 - 7 \in \mathbb{Q}[x]$, and that $\sqrt{5}$ satisfies $x^2 - 5 \in \mathbb{Q}[x]$. Hence, both number are algebraic over $\mathbb{Q}$. Furthermore, since 5 is square-free, $\sqrt{5}$ is irrational. The Gelfand-Schneider Theorem now implies that $\sqrt[3]{7}^{\sqrt{5}}$ is transcendental. □

(b) $\sqrt[3]{7}^{\sqrt{5}} + 7$ is transcendental.

Proof. Suppose by way of contradiction that $\alpha := \sqrt[3]{7}^{\sqrt{5}} + 7$ is algebraic. We apply the result of problem (3) to deduce that $\alpha - 7 = \sqrt[3]{7}^{\sqrt{5}}$ is algebraic, which contradicts part (a). □

§6.1 #11 Show that there exist irrational numbers $a, b \in \mathbb{R}$ such that a^b is rational.

Proof. We first claim that $\log_2 9$ is irrational. To see this, argue by contradiction. Suppose that $x = \log_2 9 = \frac{p}{q} \in \mathbb{Q}$. Then we have $3^2 = 9 = 2^{\log_2 9} = 2^{p/q}$. Raising both sides to the power q gives $3^{2q} = 2^p$, which contradicts unique factorization in $\mathbb{Z}$. Hence, $\log_2 9$ is irrational. Now, recall that $\sqrt{2}$ is irrational since $\sqrt{2}$ is square-free. We compute

$$\sqrt{2}^{\log_2 9} = 2^{\frac{1}{2}\cdot 2\cdot \log_2 3} = 2^{\log_2 3} = 3 \in \mathbb{Z}.$$

Here is an alternative. Consider the number $\alpha = \sqrt{2}^{\sqrt{2}}$. Since $\sqrt{2}$ is irrational, if α is rational, we have satisfied the conclusion of the problem. Suppose, on the other hand, that α is irrational. Then we have $\alpha^{\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$, in which case the conditions of the problem are also met. □