# Math 374, Final Exam Information.

**Wednesday, April 29, 2 - 5 pm, LC 405.**

## The Final Exam will be based on:

- Sections 1.1 - 1.6; 2.1 - 2.6; 3.1 - 3.4; 3.6, 4.1, 4.4.
- The corresponding assigned homework problems
  (see http://www.math.sc.edu/∼boylan/SCCourses/374Sp09/374.html).
  **At minimum, you need to understand how to do the homework problems.**

## Useful materials:

- Exams 1, 2, 3 and their solutions.
- Quizzes $1 - 9$ and their solutions.

## New Topic List (not necessarily comprehensive): (Consult review handouts for Exams I, II, III for a list of old topics.)

**You will need to know how to define vocabulary words/phrases defined in class**.

§**4.1**: **Relations:**

A relation on a set $S$ is a subset $\rho \subseteq S \times S$. We say that $a$ is related to $b$ by $\rho$ if $(a, b) \in \rho$ (we also write $a\rho b$). (More generally, one can study $n$-ary relations $\rho \subseteq S_1 \times \cdots \times S_n$.)

**Properties.**

- **Reflexivity**: $\rho$ is reflexive if and only if $\forall s \in S$, we have $(s, s) \in \rho$.
- **Symmetry**: $\forall s_1, s_2 \in S$, if $(s_1, s_2) \in \rho$, then $(s_2, s_1) \in \rho$.
- **Anti-symmetry**: $\forall s_1, s_2 \in S$ with $(s_1, s_2) \in \rho$ and $(s_2, s_1) \in \rho$, we must have $s_1 = s_2$.
- **Transitivity.** $\forall s_1, s_2, s_3 \in S$, if $(s_1, s_2) \in \rho$ and $(s_2, s_3) \in \rho$, then $(s_1, s_3) \in \rho$.

An **equivalence relation** is a relation which satisfies 1, 2, and 4 above. Let $n \geq 1$ be an integer. An important equivalence relation is $\rho_n \in \mathbb{Z} \times \mathbb{Z}$, called **congruence modulo** $n$, and defined by: $(s_1, s_2) \in \rho_n$ if and only if $s_2 - s_1$ is a multiple of $n$ (i.e., if and only if $\exists k \in \mathbb{Z}$ with $s_2 - s_1 = kn$). If $s_1 \rho_n s_2$, we also say that $s_1$ is congruent to $s_2$ modulo $n$ and write $s_1 \equiv s_2 \pmod{n}$.

If $\rho$ is an equivalence relation on $S$, an **equivalence class** of $\rho$ is a subset of elements of $S$ all of whose elements are equivalent (related) by the relation $\rho$. If $\rho$ is an equivalence relation on $S$, then the equivalence classes of $\rho$ partition the set $S$ into a disjoint union.

§**4.4**: **Functions:** Here are the basic definitions.

- Let $S$ and $T$ be sets. A rule $f : S \to T$ is a **function** if and only if $\forall s \in S$, $f$ maps $s$ to one and only one $t \in T$. I.e., $f$ is a function if and only if whenever $f(s_1) = t$ and $f(s_2) = t$, we must have $s_1 = s_2$.
- If $f : S \to T$ is a function, then $S$ is the **domain** and $T$ is the **codomain**. The **range** (or **image**) of $f$ is the set

$$f(S) = \{t \in T \ : \ \exists s \in S \text{ with } f(s) = t\}.$$

  Note that the range and the codomain are not generally the same.
- Let $s \in S$. If a function $f : S \to T$ has $f(s) = t$, then we say that $t$ is the **image** of $s$.
- Let $t \in T$ be in the range of $f : S \to T$. Then $t$ has a **preimage**: $\exists s \in S$ with $f(s) = t$. The preimage of $t$ is the set

$$f^{-1}(t) = \{s \in S \ : f(s) = t\}.$$

We gave the "ceiling" and "floor" functions as examples, and we noted that a rule $f : \mathbb{R} \to \mathbb{R}$ is a function if and only if no vertical line cuts its graph in more than one place.

**Function properties:** Let $f : S \to T$ be a function.

- $f$ is **onto** (or **surjective**) if and only if the range of $f$ is all of the codomain, $T$ (so $f(S) = T$). I.e., $f$ is onto if and only if $\forall t \in T$, $\exists s \in S$ with $f(s) = t$.
- $f$ is **one - to - one** (or **injective**) if and only if every element in the range of $f$ has one and only one preimage. I.e., $f$ is one - to - one if and only if whenever $x, y \in S$ satisfy $f(x) = f(y)$, we must have $x = y$.
- $f$ is a $1 - 1$ **correspondence** (or **bijection**) if and only if $f$ is
  1. one-to-one and
  2. onto.

  Note that if $S$ and $T$ are finite and $f : S \to T$ is a bijection, then $|S| = |T|$.

We gave examples, and noted that a function $f : \mathbb{R} \to \mathbb{R}$ is one -to-one if and only if no horizontal line cuts its graph in more than one place.

**Function composition:** Let

$$g : R \to S$$
$$f : S \to T$$

be functions. Then the composition of $g$ and $f$ is a function denoted by $g \circ f$ and satisfies

$$g \circ f : R \to T, \quad g \circ f(x) = g(f(x)).$$

We proved

**Fact.** *Function co mposition preserves injectivity and surjectivity.*

- *If $f$ and $g$ are one - to -one, then so is $g \circ f$.*
- *If $f$ and $g$ are onto, then so is $g \circ f$.*
- *If $f$ and $g$ are bijections, then so is $g \circ f$.*

**Inverse functions.**

**Identity function:** Let $R$ be a set. The **identity function** on $i_r : R \to R$ is defined by

$$i_R(r) = r, \ \ \forall r \in R.$$

Let $f : S \to T$ be a function. Then an **inverse of** $f$ (if it exists) is a function $g : T \to S$ with

$$f \circ g = i_T$$
$$g \circ f = i_S.$$

If such a $g$ exists, we write $g = f^{-1}$. Not every function has an inverse. In this direction, we proved:

**Fact.** *The function $f : S \to T$ has an inverse if and only if it is a bijection.*

We gave some examples and non-examples. We proved the following additional related fact:

**Fact.** *If $f$ and $g$ are bijections, then so is $g \circ f$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

**Order of magnitude of functions.**

Let $f$ and $g$ be functions mapping non-negative reals to non-negative reals:

$$f, g : \{x \in \mathbb{R} : x \geq 0\} \to \{x \in \mathbb{R} : x \geq 0\}.$$

- We say that $f$ is the same **order of magnitude** as $g$, written $f = \Theta(g)$, if there are positive constants $n_0$, $c_1$, and $c_2$ such that $\forall x \geq n_0$, we have

$$c_1 \cdot g(x) \leq f(x) \leq c_2 \cdot g(x).$$

- We say the $f$ is **big oh** of $g$, written $f = O(g)$ if there are positive constants $n_0$ and $c$ such that $\forall x \geq n_0$,

$$f(x) \leq cg(x).$$

  This says that $f$ grows at the same rate or at a slower rate than $g$.