

Math 374, Exam 3 Information

4/17/09, LC 405, 2:30 - 3:20

Exam 3 will be based on:

- Sections 3.1 - 3.4, 3.6, and 4.1.
- The corresponding assigned homework problems
(see <http://www.math.sc.edu/~boylan/SCCourses/374Sp09/374.html>)
At minimum, you need to understand how to do the homework problems.
- Lecture notes: 3/16 - 4/10.

Topic List (not necessarily comprehensive):

You will need to know how to define vocabulary words/phrases defined in class.

§3.1: Sets. Terms and objects from this section: Cardinality (size of a set), the empty set (\emptyset : the set with no elements), subsets (proper and improper subsets: if a set $S \neq \emptyset$, then its only improper subsets are \emptyset and S (itself): all other subsets are proper), power set of a set S ($\wp(S)$: the set of all subsets of S ; if S is finite, then $|\wp(S)| = 2^{|S|}$), set operations (union, intersection, set difference, complement, cross product), Venn diagrams (as a way to represent relationships between sets). We discussed how to prove identities for expressions involving sets by using set algebra (definitions and properties of the basic set operations) and double-set inclusion: one way to prove that $A = B$ is to first show that $A \subseteq B$, then to show that $B \subseteq A$.

A set S is countable if it is either finite, or if it can be put into a one-to-one correspondence with the set \mathbb{N} of natural numbers. Examples of infinite countable sets include \mathbb{Z} (integers) and \mathbb{Q} (rationals). The set \mathbb{R} of real numbers is uncountable.

§3.2: Counting. Multiplication principle: If there are n_1 possible outcomes for event A and n_2 possible outcomes for event B , then the number of outcomes for A and B ($A \cap B$) is $n_1 n_2$. **Addition principle:** If there are n_1 possible outcomes for event A and n_2 possible outcomes for event B , and if A and B are disjoint ($A \cap B = \emptyset$), then the number of outcomes for A or B ($A \cup B$) is $n_1 + n_2$.

§3.3: Principle of inclusion and exclusion; pigeonhole principle. Basic versions of the principle of inclusion and exclusion (PIE):

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|.$$

The general version gives a formula for $|A_1 \cup A_2 \cup \cdots \cup A_n|$.

Pigeonhole principle: If $k + 1$ (or more) objects are placed into k boxes, then there is at least one box containing 2 or more objects.

§3.4: Permutations and combinations. If S is a set of size n and $0 \leq r \leq n$, then an r -permutation is an **ordered** arrangement of any r elements from S . The number of r -permutations from a set of n objects is

$$P(n, r) = \frac{n!}{(n - r)!}.$$

Note that $0! = 1$, so $P(n, n) = n!$ and $P(n, 0) = 1$.

An r -combination of elements of a set S is any **unordered** selection of r elements from S . Equivalently, an r combination is a subset of S of size r . Then number of r -combinations from a set of size n is

$$C(n, r) = \frac{n!}{r!(n - r)!} = \frac{P(n, r)}{r!}.$$

Sometimes we write $\binom{n}{r}$ (read: "n choose r") to denote $C(n, r)$. $C(n, r)$ satisfies lots of (useful) identities. One such identity is

$$C(n, r) = C(n, n - r).$$

More identities were discussed in §3.6.

Permutations with repetitions:

- The number of r -permutations of a set S of size n where each element of S can be used in the permutation an **unlimited** number of times is n^r .
- Suppose that a set S of size n consists of n_1 objects of type 1, n_2 objects of type 2, ..., n_k objects of type k (so $n_1 + n_2 + \cdots + n_k = n$). Then the number of permutations of elements in the set S is

$$\frac{n!}{n_1!n_2! \cdots n_k!}.$$

Example: How many permutations are there of the letters in "Mississippi"?

Combinations with repetitions: If a set S has n types of objects, then the number of r -combinations (with unlimited repetitions) of elements from the set S is

$$\binom{r + n - 1}{r} = C(r + n - 1, r).$$

Example: Suppose that you choose 8 donuts from a selection of 5 types of donuts. (You may choose as many donuts as you want (up to 8) of a given type of donut.) How many ways can this be done?

§3.6: Binomial theorem. Let $n \geq 0$ be an integer. Then we have

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i} = \sum_{i=0}^n C(n, i) x^i y^{n-i}.$$

One can prove lots of identities involving the symbols $C(n, r) = \binom{n}{r}$ using either of two methods:

- **Algebra.** Use the binomial theorem to show that both sides of the identity are equal.
- **Counting.** Show that both sides of the identity count the same objects.

Examples of this include:

$$\sum_{k=0}^n \binom{n}{k} = 2^n, \quad \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}, \quad \sum_{k=1}^n k \binom{n}{k} = n \cdot 2^{n-1}.$$

The multinomial symbol is defined as follows: For $1 \leq i \leq t$, suppose that n_i is a non-negative integer, and suppose that $n_1 + \cdots + n_t = n$. Then

$$\binom{n}{n_1 \ n_2 \ \cdots \ n_t} = \frac{n!}{n_1! \cdots n_t!}.$$

For non-negative integers n , the multinomial theorem is given by

$$(x_1 + \cdots + x_t)^n = \sum_{\substack{n_1, \dots, n_t \geq 0 \\ n_1 + \cdots + n_t = n}} \binom{n}{n_1 \ \cdots \ n_t} x_1^{n_1} \cdots x_t^{n_t}.$$

§4.1: Relations. A relation on a set S is a subset $\rho \subseteq S \times S$. We say that a is related to b by ρ if $(a, b) \in \rho$ (we also write $a\rho b$). (More generally, one can study n -ary relations $\rho \subseteq S_1 \times \cdots \times S_n$.)

Properties.

- **Reflexivity:** ρ is reflexive if and only if $\forall s \in S$, we have $(s, s) \in \rho$.
- **Symmetry:** $\forall s_1, s_2 \in S$, if $(s_1, s_2) \in \rho$, then $(s_2, s_1) \in \rho$.
- **Anti-symmetry:** $\forall s_1, s_2 \in S$ with $(s_1, s_2) \in \rho$ and (s_2, s_1) , we must have $s_1 = s_2$.
- **Transitivity.** $\forall s_1, s_2, s_3 \in S$, if $(s_1, s_2) \in \rho$ and $(s_2, s_3) \in \rho$, then $(s_1, s_3) \in \rho$.

An **equivalence relation** is a relation which satisfies 1, 2, and 4 above. Let $n \geq 1$ be an integer. An important equivalence relation is $\rho_n \in \mathbb{Z} \times \mathbb{Z}$, called **congruence modulo n** , and defined by: $(s_1, s_2) \in \rho_n$ if and only if $s_2 - s_1$ is a multiple of n (i.e., if and only if $\exists k \in \mathbb{Z}$ with $s_2 - s_1 = kn$). If $s_1 \rho_n s_2$, we also say that s_1 is congruent to s_2 modulo n and write $s_1 \equiv s_2 \pmod{n}$.

If ρ is an equivalence relation on S , an **equivalence class** of ρ is a subset of elements of S all of whose elements are equivalent (related) by the relation ρ . If ρ is an equivalence relation on S , then the equivalence classes of ρ partition the set S into a disjoint union.