Math 241, Exam 3 Information.

4/15/09, LC 115, 12:20 - 1:10.

Exam 3 will be based on:

- Sections 14.7 - 14.9, 15.1 - 15.3.
- The corresponding assigned homework problems
  (see http://www.math.sc.edu/∼boylan/SCCourses/241Sp09/241.html)
  At minimum, you need to understand how to do the homework problems.
- Lecture notes: 3/16 - 4/10.

Topic List (not necessarily comprehensive):

You will need to know how to define vocabulary words/phrases defined in class.

§**14.7**: **Tangent planes and normal vectors**.

1. Suppose that the surface has equation $z = f(x, y)$ and that $P(x_0, y_0)$ is a point on the surface. Then the tangent plane at P is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0.$$

   You can rearrange it (as in the book) to obtain

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

   From either equation, you can see that a normal vector to the tangent plane is

$$\vec{\mathbf{n}} =< f_x(x_0, y_0), f_y(x_0, y_0), -1 > .$$

   The normal line through $P(x_0, y_0)$ has direction vector $\vec{\mathbf{n}}$ and parametric equations

$$\begin{aligned} x &= x_0 + f_x(x_0, y_0)t \\ y &= y_0 + f_y(x_0, y_0)t \\ z &= f(x_0, y_0) - t. \end{aligned}$$

2. Suppose that the surface has equation $F(x, y, z) = 0$ and that $P(x_0, y_0, z_0)$ is a point on the surface. The tangent plane at P is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

   A normal vector to the tangent plane at P is simply **the gradient vector** of F at P:

$$\vec{\mathbf{n}} = \nabla F =< F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0) >$$

   The normal line through the point P has direction vector $\vec{\mathbf{n}}$ and parametric equations:

$$\begin{aligned} x &= x_0 + F_x(x_0, y_0, z_0)t \\ y &= y_0 + F_y(x_0, y_0, z_0)t \\ z &= z_0 + F_z(x_0, y_0, z_0)t. \end{aligned}$$

Do not forget the basic information from §12 on lines, planes, and vectors!

§**14.8**: **Maxima and minima of functions of two variables**. How do you find **relative extrema**? Suppose that $z = f(x, y)$. The same idea works for $z = f(x, y, z)$.

1. Determine the **critical points** of f. These are the points $P(x_0, y_0)$ for which $f_x(x_0, y_0) = 0$ **and** $f_y(x_0, y_0) = 0$. (To find such P, you will solve a system of two equations in the variables x and y.) The critical points are the "candidates" for relative extrema.
2. Next, apply the **Second Partials Test** (see Theorem 14.8.6) to each critical point that you found in step 1. The test will determine whether the critical point is a relative max., relative min., or saddle; it may also happen that the test yields no information.

How do you find absolute extrema? To find absolute extrema of $z = f(x, y)$ on a closed and bounded subset R of $\mathbb{R}^2$ (the same ideas apply to functions of 3 vars.), proceed as follows:

1. Draw R, if possible, and find the **critical points** of the function f that lie in the interior of R (i.e., inside R, but not on its boundary). The method for finding critical points is the same as in in step 1 above.
2. Find all **boundary points** at which absolute extrema could occur. These include endpoints of the pieces of the boundary. There could be more than these, however. To find all such points, write down equations for the boundary of R (in terms of the variables x and y; these are typically equations of lines and circles). For each piece of the boundary, substitute its equation back into the original equation $f(x, y)$. Doing so should eliminate either x or y from $f(x, y)$; what remains should be a function $u(x)$ or $v(y)$ of x or y alone. Now find the critical points for the single variable function $u(x)$ or $v(y)$.
3. Test all of the "candidate" points $P(x_0, y_0)$ obtained from steps 1 and 2 by plugging them into the original equation $f(x, y)$. The largest value is the absolute max; the smallest is the absolute min.

§**14.9**: **Lagrange multipliers**. Find the absolute max. or min. of a function $z = f(x, y)$ subject to the constraint $g(x, y) = 0$. (Again, the same ideas carry over to functions of 3 variables). Candidates for points $P(x_0, y_0)$ at which absolute extrema could occur have $g(x_0, y_0) = 0$ **and** $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some $\lambda \neq 0$. I.e., absolute extrema must occur at points for which the gradients of f and g are parallel. The constant λ is the "Lagrange multiplier". To find such P, we proceed as follows:

1. Compute ∇f, ∇g.
2. Set $\nabla f = \lambda \nabla g$. Our functions have two variables, so this translates into two equations. Solve for λ in both equations. This gives two expressions for λ in terms of the variables x and y. The equality of these two expressions gives a relationship between x and y.
3. Substitute the relationship between x and y into the constraint equation $g(x, y) = 0$. Typically, this eliminates one of x or y so that you can solve for the other.

Once you have obtained "candidate" points P, you plug them into the function $f(x, y)$ to see which yields an absolute max. or min.

§**15.1**: **Double integrals**. What is a double, or iterated integral? In this section, the integration is done over rectangular regions, so all limits of integration are constant. In this setting we have:

$$\int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx.$$

Switching the order of integration is only this easy when you are integrating over rectangles!

§**15.2**: **Double over non-rectangular regions**. To integrate $\iint\limits_R f(x,y)dA$, one may view R as a type I or type II region.

1. Viewing R as a type I region, we have

$$\iint\limits_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

2. Viewing R as a type II region, we have

$$\iint\limits_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

When the region R is not rectangular, switching the integration order will change the limits. To determine the new limits, it usually helps to draw R.

- **Volume**: The volume above a region R in the xy-plane and below $z = f(x,y) \geq 0$ is

$$V = \iint\limits_R f(x,y)dA.$$

- **Area**: The area enclosed by a region R in the xy-plane is

$$A = \iint\limits_R 1dA.$$

§**15.3**: **Double integrals in polar coordinates**. A point P has polar coordinates (r,θ) if and only if r is the distance from P to the origin O and the ray OP makes an angle θ with the positive x-axis. Here is how polar and rectangular coordinates are related:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

$$r^2 = x^2 + y^2 \quad (\text{so } r = \sqrt{x^2+y^2}), \qquad \tan\theta = \frac{y}{x} \quad (\text{so } \theta = \tan^{-1}\left(\frac{y}{x}\right)).$$

Here is how double integrals are related in the two coordinate systems:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta.$$