Math 241. Exam 2. 3/18/09. **Name:** ________________

- Read problems carefully. Show all work.
- No notes.
- The exam is approximately 15 percent of the total grade.
- There are 100 points total. Partial credit may be given.

1. **(15 points)** Suppose that a curve is parametrized by

$$\vec{r}(t) = (1 + 2\cos t)\vec{i} + (-1 + 2\sin t)\vec{j}.$$

(a) **(10 points)** Find an arc-length parametrization for the curve with reference point $(3, -1)$. (Note that this corresponds to the value $t = 0$).

$$S = \int_0^t \left((-2\sin\tau)^2 + (2\cos\tau)^2 \right)^{1/2} d\tau$$

$$= \int_0^t \left(4\underbrace{(\sin^2\tau + \cos^2\tau)}_{=1} \right)^{1/2} d\tau = \int_0^t 2\, d\tau = 2t$$

$$\Rightarrow t = S/2 \Rightarrow \boxed{\vec{r}(s) = \left(1 + 2\cos\left(\tfrac{s}{2}\right)\right)\vec{i} + \left(-1 + 2\sin\left(\tfrac{s}{2}\right)\right)\vec{j}}$$

(b) **(5 points)** Find a point π units from the reference point along the curve.

$$\vec{r}(\pi) = \left(1 + 2\cos\left(\tfrac{\pi}{2}\right)\right)\vec{i} + \left(-1 + 2\sin\left(\tfrac{\pi}{2}\right)\right)\vec{j}$$

$\Rightarrow$ the point is $\boxed{(1, 1)}$

2. **(5 points)** Compute the general limit, if it exists. **Show** how you obtained your answer.

$$\lim_{(x,y)\to(-1,1)} \frac{x^2y^2-1}{xy+1}.$$

$$= \lim_{(x,y)\to(-1,1)} \frac{(xy-1)(xy+1)}{xy+1} = \lim_{(x,y)\to(-1,1)} (xy+1) = \boxed{0}$$

$$\lim_{(x,y)\to(1,1)} \frac{x^2y^2-1}{xy-1}$$

$$= \lim_{(x,y)\to(1,1)} \frac{(xy-1)(xy+1)}{xy-1} = \lim_{(x,y)\to(1,1)} (xy+1) = \boxed{2}$$

3. **(10 points)** Let $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$.

(a) **(4 points)** Find the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

along the line $y = x$, if it exists.

Along $y = x$, $f(x,y) = \dfrac{x^2 \cdot x}{x^4 + x^2} = \dfrac{x^3}{x^2(x^2+1)} = \dfrac{x}{x^2+1}$

$\Rightarrow$ the limit is $\lim\limits_{x\to 0} \dfrac{x}{x^2+1} = \dfrac{0}{1} = \boxed{0}$

(b) **(4 points)** Find the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

along the parabola $y = x^2$, if it exists.

Along $y = x^2$, $f(x,y) = \dfrac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \dfrac{x^4}{2x^4} = \dfrac{1}{2}$

$\Rightarrow$ the limit is $\boxed{\dfrac{1}{2}}$

(c) **(2 points)** What, if anything, can you conclude about the general limit,

$$\lim_{(x,y)\to(0,0)} f(x,y),$$

from the parts (a) and (b)?

$0 \neq \frac{1}{2} \Rightarrow$ the limit does not exist

4. **(20 points)** Let $f(x,y) = \sin(x^2+y^2) + e^{y^5}x^3$.

(a) **(6 points)** Compute f_x.

$$f_x = \cos(x^2+y^2)\cdot 2x + 3x^2 e^{y^5}$$

$$= \boxed{2x\cos(x^2+y^2) + 3x^2 e^{y^5}}$$

(b) **(8 points)** Compute $\dfrac{\partial^2 f}{\partial y \partial x}$. You may use part (a) to do this.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(2x\cos(x^2+y^2) + 3x^2 e^{y^5}\right)$$

$$= 2x\left(-\sin(x^2+y^2)\cdot 2y\right) + 3x^2\cdot e^{y^5}\cdot 5y^4$$

$$= \boxed{-4xy\sin(x^2+y^2) + 15x^2y^4e^{y^5}}$$

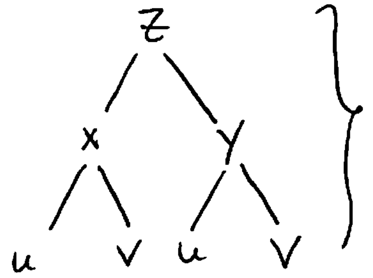
(c) **(6 points)** What is $\dfrac{\partial^2 f}{\partial x \partial y}$? You may use previous parts of this problem, if necessary, to do this.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \boxed{-4xy\sin(x^2+y^2) + 15x^2y^4e^{y^5}}$$

5. **(20 points)**

(a) **(10 points)** Let $z = x\cos y$, $x = \ln(u^2 + uv)$, and $y = uv^2$.

Use the **chain rule** to find $\dfrac{\partial z}{\partial u}$. Your answer should be in one or both of the variables u and v.



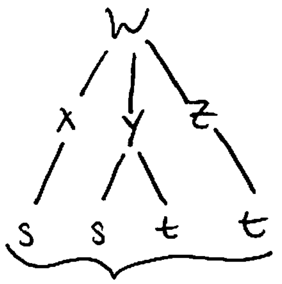
$$\Longrightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= (\cos y)\left(\frac{1}{u^2+uv}\right)(2u+v) + (-\sin y)\, x \cdot v^2$$

$$= \boxed{\cos(uv^2)\left(\frac{2u+v}{u^2+uv}\right) - v^2 \sin(uv^2)\ln(u^2+uv)}$$

(b) **(10 points)** Let $w = 3xyz,\ x = 1/s^3,\ y = s^2 t \sin t,\ z = e^{-s}/s$.

i. **(5 points)** Write down the **form** of the chain rule that you would use to compute $\dfrac{\partial w}{\partial s}$. (I am only asking for the rule, **not** a computation).



$$\boxed{\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{dz}{ds}}$$

ii. **(5 points)** Write down the **form** of the chain rule that you would use to compute $\dfrac{\partial w}{\partial t}$. (I am only asking for the rule, **not** a computation).

$$\boxed{\frac{\partial w}{\partial t} = \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}}$$

6. **(20 points)** Let $f(x,y,z) = 4x^2 - 5xy + xyz$.

(a) **(10 points)** Find the rate of change of f (the directional derivative) at $P(1,1,1)$ in the direction of the vector $\vec{\mathrm{v}} = \vec{\mathrm{i}} + \vec{\mathrm{j}} - \vec{\mathrm{k}}$.

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{i}+\vec{j}-\vec{k}}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}\right)\vec{i} + \left(\frac{1}{\sqrt{3}}\right)\vec{j} + \left(-\frac{1}{\sqrt{3}}\right)\vec{k}$$

$$f_x = 8x - 5y + yz \Rightarrow f_x(P) = 8-5+1 = 4$$

$$f_y = -5x + xz \Rightarrow f_y(P) = -5+1 = -4$$

$$f_z = xy \Rightarrow f_z(P) = 1.$$

$$D_{\vec{u}} f(P) = \frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \boxed{-\frac{1}{\sqrt{3}}}$$

(b) **(5 points)** Give a **unit vector** in the direction in which f changes most rapidly at P.

$$\frac{\nabla f(P)}{\|\nabla f(P)\|} = \boxed{\frac{4\vec{i} - 4\vec{j} + \vec{k}}{\sqrt{33}}}$$

(c) **(5 points)** What is the **maximum** rate of change of f at P?

$$\|\nabla f(P)\| = \sqrt{33}.$$

7. **(10 points)** Suppose that the elevation of a mountain above sea level at the point (x, y) is $z = f(x, y)$. A climber at a point $P(x_0, y_0)$ on the mountain sees that the slope of the mountain in the eastern direction is -3 (so $f_x(x_0, y_0) = -3$) and that the slope in the northern direction is -1 (so $f_y(x_0, y_0) = -1$).

   (a) **(5 points)** What is $\nabla f(x_0, y_0)$, the gradient vector at P?

$$\boxed{\nabla f(P) = -3\vec{i} - \vec{j}}$$

   (b) **(5 points)** Give a **unit vector** in the direction of fastest **descent** at P down the mountain (the direction of maximum rate of **decrease** of $f(x, y)$ at P).

$$-\frac{\nabla f(P)}{\|\nabla f(P)\|} = \boxed{\frac{3\vec{i} + \vec{j}}{\sqrt{10}}}$$