Math 241. Exam 1. 2/11/09. **Name:** ________________

- Read problems carefully. Show all work.
- No notes or calculators.
- The exam is approximately 15 percent of the total grade.
- There are 100 points total. Partial credit may be given.

1. **(10 points)** Find an equation for the sphere with a diameter whose endpoints are $(3, 2, -3)$ and $(1, -4, 5)$.

Center = midpt of endpts = $(2, -1, 1)$

radius = distance from $(3,2,-3)$ to $(2,-1,1)$: $\sqrt{(3-2)^2 + (2-(-1))^2 + (-3-1)^2}$

$= \sqrt{1+9+16} = \sqrt{26}.$

Eq: $\boxed{(x-2)^2 + (y+1)^2 + (z-1)^2 = 26}$

2. **(10 points)** Find a vector of length 2 in the direction opposite to the vector $\vec{u} = \vec{AB}$ from point $A(2, 0, -1)$ to point $B(-1, 2, 3)$.

$\vec{u} = \langle -3, 2, 4 \rangle$. $(-2)\dfrac{\vec{u}}{\|\vec{u}\|} = -2\dfrac{\langle -3, 2, 4\rangle}{\sqrt{9+4+16}} = \boxed{\dfrac{-2}{\sqrt{29}}\langle -3, 2, 4\rangle}$

3. **(20 points)** Consider the vectors

$$\vec{u} = 2\vec{i} - \vec{k}, \quad \vec{v} = -3\vec{i} + \vec{j} + 2\vec{k}.$$

(a) **(10 points)** Let θ be the angle between the vectors $\vec{u}$ and $\vec{v}$. What is $\cos\theta$?

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{-6-2}{(\sqrt{4+1})(\sqrt{9+1+4})} = \boxed{\frac{-8}{\sqrt{70}}}$$

(b) **(10 points)** Find the area of a parallelogram that has adjacent sides $\vec{u}$ and $\vec{v}$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ -3 & 1 & 2 \end{vmatrix} = \vec{i}(0-(-1)) - \vec{j}(4-3) + \vec{k}(2)$$

$$= \vec{i} - \vec{j} + 2\vec{k}$$

$$\text{Area} = \sqrt{1+1+4} = \boxed{\sqrt{6}} = \|\vec{u}\times\vec{v}\|$$

4. **(15 points)** No justification required. Let $\vec{u}$ and $\vec{v}$ be **non-zero** vectors.

(a) **(5 points)** Circle the statement(s) which are true **for all** $\vec{u}$ and $\vec{v}$.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$. $\leftarrow$ False!

(b) **(5 points)** Circle the correct answer. If $\vec{u} \cdot \vec{v} = 0$, then $\vec{u}$ and $\vec{v}$ must be

- **parallel**
- **orthogonal**
- possibly **neither**.

(c) **(5 points)** Circle the correct answer. If $\vec{u} \times \vec{v} = 0$, then $\vec{u}$ and $\vec{v}$ must be

- **parallel**
- **orthogonal**
- possibly **neither**.

5. **(10 points)** Determine whether the following line L and plane Q are parallel, perpendicular, or neither. Briefly explain your answer.

$$L: \begin{aligned} x &= 1 + 2t \\ y &= 2 + 3t \\ z &= -1 + 2t. \end{aligned}$$

$$Q: \ 5x - 4y + z = 3.$$

$\vec{V} = \langle 2, 3, 2 \rangle$ (dir. vector for L)

$\vec{n} = \langle 5, -4, 1 \rangle$ (normal vector to Q)

$\vec{n} \cdot \vec{V} = 10 - 12 + 2 = 0 \Longrightarrow \vec{n} \perp \vec{V} \Rightarrow \boxed{L \parallel Q \text{ (parallel)}}$

6. **(15 points)** Find the standard equation of the plane through the point $(1,-2,3)$ that is perpendicular to the planes

$$Q_1 : x - 2y + z = -4$$
$$Q_2 : 3x - y - z = 2.$$

Show your work.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 3 & -1 & -1 \end{vmatrix} = \vec{i}(2-(-1)) - \vec{j}(-1-3) + \vec{k}(-1-(-6))$$
$$= 3\vec{i} + 4\vec{j} + 5\vec{k}$$

Eq: $\boxed{3(x-1) + 4(y+2) + 5(z-3) = 0}$

or $\boxed{3x + 4y + 5z = 10} \leftarrow$ (standard form)

7. **(20 points)**

(a) **(13 points)** Find an equation of the line tangent to the graph of

$$\vec{r}(t) = (t^3 - t)\vec{i} + (6t+1)\vec{j} - (t^4)\vec{k}$$

at the point where $t = 1$.

$\vec{r}(1) = 7\vec{j} - \vec{k} \Rightarrow$ pt. of tangency is $(0, 7, -1)$

$\vec{r}\,'(t) = (3t^2 - 1)\vec{i} + 6\vec{j} - 4t^3\vec{k}$

$\Rightarrow \vec{r}\,'(1) = 2\vec{i} + 6\vec{j} - 4\vec{k}$ = tan. line ~~dir.~~ dir. vector

$\Rightarrow$ Eq: $\boxed{\begin{aligned} x &= 0 + 2t \\ y &= 7 + 6t \\ z &= -1 - 4t. \end{aligned}}$

(b) **(7 points)** Is the tangent line in part (a) parallel to the line with vector equation

$$r(\vec{t}) = (1+4t)\vec{i} + (-2+12t)\vec{j} + (7-t)\vec{k}?$$

Why or why not? Briefly explain

No. It has dir vector $\langle 4, 12, -1\rangle \neq k\underbrace{\langle 2, 6, -4\rangle}_{\text{dir. vector of tan line}}$ for any real k.