Math 241, Final Exam Information.

Saturday, May 2, 2 - 5 pm, LC 115.

Review: April 30, 6 pm, LC 112.

The Final Exam will be based on:

- Sections 12.1 - 12.6, 13.1 - 13.3, 14.1 - 14.3, 14.5 - 14.9, 15.1 - 15.3, 15.5, 15.7, 15.8.
- The corresponding assigned homework problems (see http://www.math.sc.edu/∼boylan/SCCourses/241Sp09/241.html). **At minimum, you need to understand how to do the homework problems.**

Useful materials:

- Exams 1, 2, 3 and their solutions.
- Quizzes $1 - 9$ and their solutions.

New Topic List (not necessarily comprehensive): (Consult review handouts for Exams I, II, III for a list of old topics.)

You will need to know how to define vocabulary words/phrases defined in class.

§**15.5**: **Triple integrals:** Be able to set up and evaluate triple integrals in rectangular coordinates:

$$\iiint\limits_G f\,dV = \iint\limits_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right) dA$$

Note that the limits on the inner integral are functions of (at most) the outer variables x and y. We typically try to view G as a simple xy-solid. In particular, we try to identify:

- The "top" of G: $z = g_2(x,y)$; the "bottom" of G: $z = g_1(x,y)$. We must have $g_1(x,y) \leq g_2(x,y)$ for all x, y.
- The projection R (or "shadow") of G on the xy-plane. We try to find equations for the boundary of R.

To set up the limits, you may need to do two separate sketches: a sketch of the solid G (in xyz-space), and a sketch of the projection R (in the xy-plane). Once we evaluate the inner integral, a double integral remains, which we must integrate:

$$\iint\limits_R F(x,y)\,dA.$$

This can be done by viewing R as a type I region (integrate with respect to y first) or as a type II region (integrate with respect to x first) as in section §15.2.

As a triple integral, the volume of a solid G is

$$V = \iiint\limits_G dV.$$

§**15.7**: **Triple integrals in cylindrical and spherical coordinates:**

- **Cylindrical coordinates:** The cylindrical coordinates are (r, θ, z). These should be viewed as extending polar coordinates (r, θ) by "throwing in" the rectangular height coordinate z. The relevant formulas are these:

  $$z = z, \quad x = r\cos\theta, \quad y = r\sin\theta;$$
  $$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

  Further, the volume element, dV, is given as

  $$dV = r\,dz\,dr\,d\theta.$$

  Triple integrals in cylindrical coordinates take the form

  $$\iiint\limits_G f dV = \iint\limits_R \left(\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)dz\right) dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

  Note that the order of integration is z, r, θ. As for integrals in rectangular coordinates, you will need to identify:

  - The "top" of G: $z = g_2(r,\theta)$; the "bottom" of G: $z = g_1(r,\theta)$. The difference is that now, the equations for the top and bottom are in the polar variables r and θ.
  - The projection R (or "shadow") of G on the xy-plane. We try to find equations for the boundary of R **viewed as a polar region**.

  As in the case of rectangular coordinates, it is helpful to do a sketch of G and a sketch of R.

- **Spherical coordinates:** The spherical coordinates are (ρ, θ, ϕ). The relevant formulas are these:

  $$z = \rho\cos\phi, \quad x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta;$$
  $$\rho = \sqrt{x^2+y^2+z^2}.$$

  Further, the volume element, dV, is given as

  $$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

  Triple integrals in spherical coordinates take the form

  $$\iiint\limits_G f dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Note that the order of integration is ρ, ϕ, θ. As in the case of other coordinate systems, it is helpful to do a sketch of the solid G to determine the limits. In this case, however, we do not need to sketch R, the projection, since the triple integral does not involve R.

§**15.8**: **Change of variable in multiple integrals; Jacobians:**

The **Jacobian** is a necessary ingredient in the change variables formula for multiple integrals. Suppose that a change of variables is given by

$$x = x(u, v), \quad y = y(u, v).$$

(so we want to change from (x, y) to (u, v)). The Jacobian of this change of variables is defined by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}.$$

Now, the change of variables formula for double integrals is

$$\iint\limits_R f(x, y)\, dy\, dx = \iint\limits_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dv\, du.$$

In particular, note that

- The new integrand is $f(x(u, v), y(u, v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$.
- $dy\, dx$ becomes $dv\, du$.
- The region over which the integration is to be done changes from the region R in the xy-plane to the region S in the uv-plane.

The third bullet tells us how the x- and y-limits change to u- and v-limits. Specifically, the change of variables will take the boundary of R to the boundary of S. Noting this, it is usually not hard to determine S, and therefore, the u- and v- limits. For this purpose, you will probably need to sketch both R and S.

One can appropriately specialize the change of variables formula for double integrals to single integrals, giving the usual "u-substitution" from first-semester 1-variable calculus. Similarly, one can appropriately generalize the formula to triple, quadruple, etc. integrals.

An important change of variables for double integrals is given by the polar coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \frac{\partial(x, y)}{\partial(r, \theta)} = r,$$

which yields

$$\iint\limits_{R \text{ rectangular}} f(x, y)\, dy\, dx = \iint\limits_{S \text{ polar}} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

We studied this in §15.3. Further, a computation shows that the Jacobians for cylindrical and spherical coordinates are

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r, \quad \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \rho^2 \sin\phi.$$

We also know this from our work in §15.7.