Mathematical Imaging

Francisco Blanco-Silva

April 3, 2008

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University of South Carolina

Mathematicians are like Frenchmen: Whatever you say to them, they translate into their own language and forthwith it is something entirely different.

Johan Wolfgang von Goethe

Outline

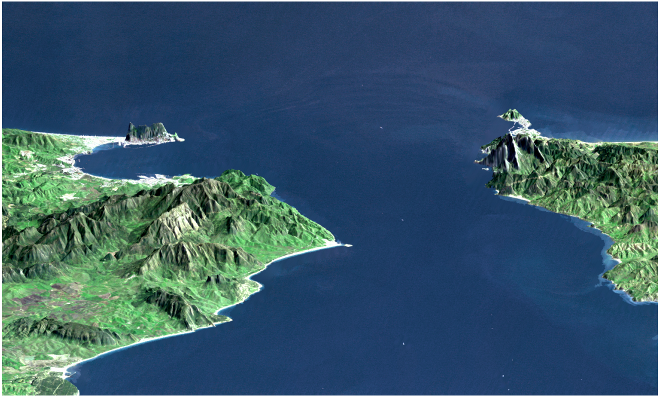
Image Acquisition
- Optical Imaging
- Imaging from Wave Propagation
- Other Imaging Devices

Image Processing
- Image Analysis
- Editing and Restoration
- Image Compression

Techniques
- Learning Theory
- P.D.E. Models
- Inverse Problems
- Approximation Theory
- Multi-resolution Analysis
- Variational Methods

An Example

Image generated from a Landsat satellite image draped over an elevation model produced by the Shuttle Radar Topography Mission (SRTM).
Natural colors of the scene are enhanced by image processing. The scene includes some infrared reflectance (as green) to highlight the vegetation pattern as well as shading of the elevation model to further highlight the topographic features.



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Optical Imaging

- (Chemical) Photography/Film
- Computational Imaging (Digital Photography/Video)
- Dark-field Imaging (Scattered-light registration)

L. J. M. Daguerre.
First photography, taken in 1839

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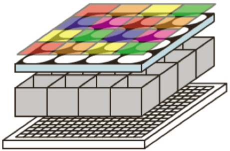

Fig. 3. TOMBO system with different interference filter on each unit.

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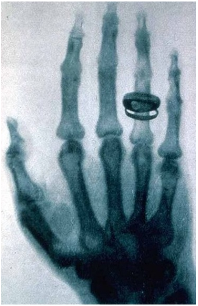
Imaging from Wave Propagation

- Transmission Imaging
- Radar Imaging
- Interferometry
- Quantum Tunneling
- Volume Imaging



The hand of Mrs. Wilhelm Röntgen: The first X-ray image, 1895.

In Otto Glasser, "*Wilhelm Conrad Röntgen and the early history of the Röntgen rays.*" London, 1933. National Library of Medicine.

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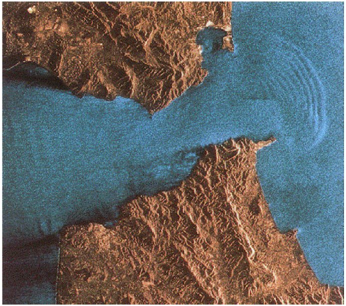

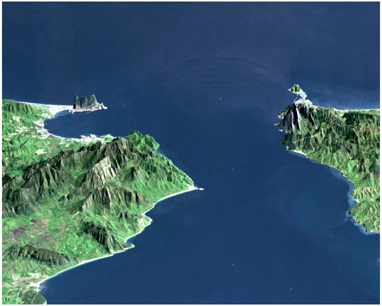

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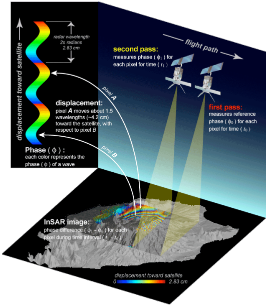

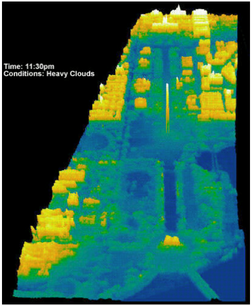

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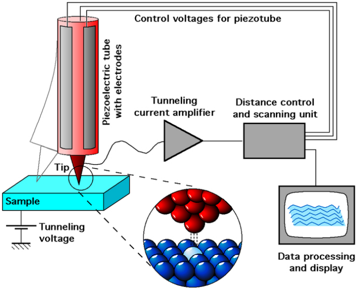

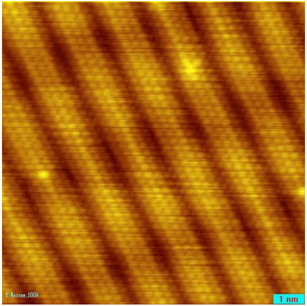

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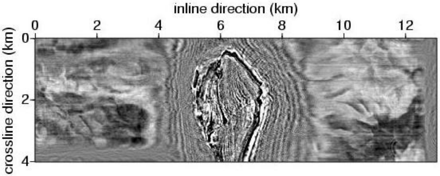

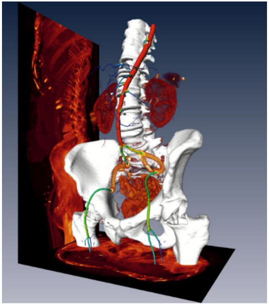

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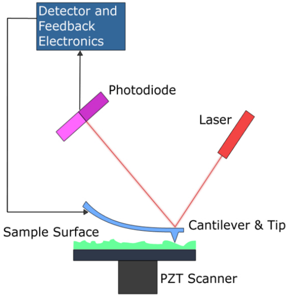
Image from Wave Propagation

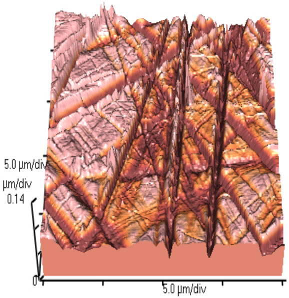
Volume Imaging

- Atom probe tomography
- Confocal laser scannig microscopy
- Cryo-electron tomography
- Electrical capacitance tomography
- Electrical impedance tomography
- Functional magnetic resonance imaging
- Magnetic induction tomography
- Magnetic resonance imaging
- Optical projection tomography
- Photoacoustic tomography
- Positron emission tomography
- Quantum tomography
- Single photon emission computed tomography
- Seismic tomography
- Ultrasound transmission tomography
- X-ray tomography
- Zeeman-Doppler

Other Imaging Devices

- Atomic-force Imaging
- Printing





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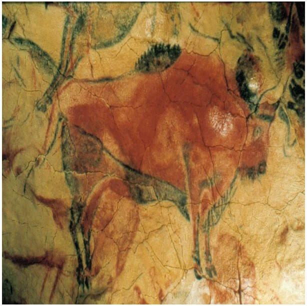

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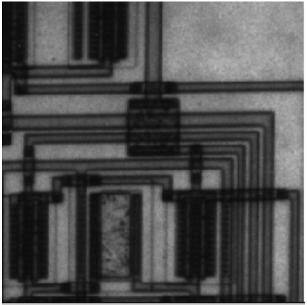
Image Analysis

Low-level Image Analysis

Low level analysis aims at extracting reliable, local geometric information from a mathematical image:

- Edge detection
- Level lines, curvature.



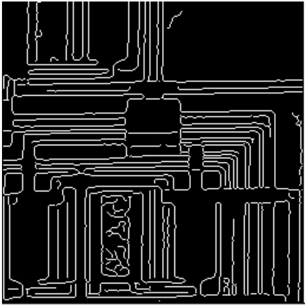


Image Analysis

Global Image Analysis

- Smoothness
- Quadtree decomposition
- Perimeter Determination
- Image Area
- Topology (Euler number)

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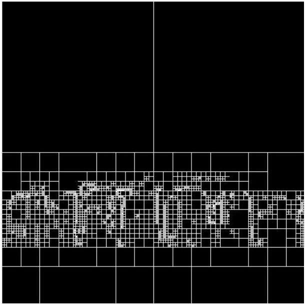

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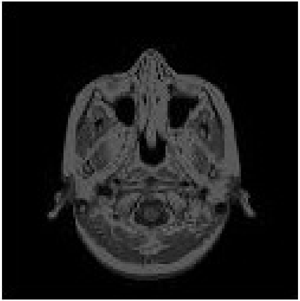
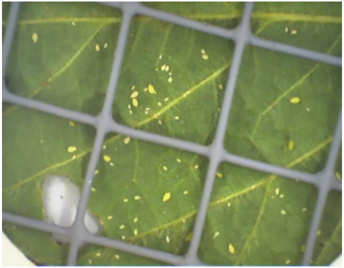


Image Analysis

Visual Learning and Recognition

- Segmentation
- Pattern Recognition
- Motion Recognition
- Data Cloud Assimilation



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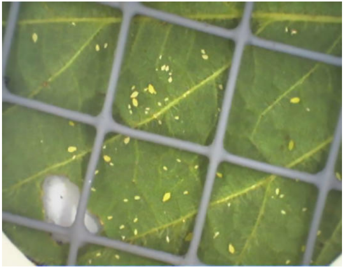

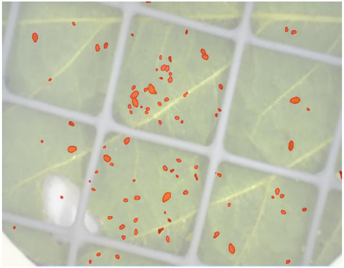

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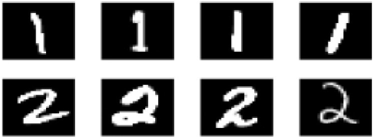

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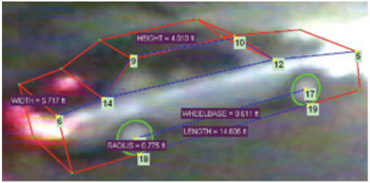

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AFRL/MNG VEAA Data Set#1

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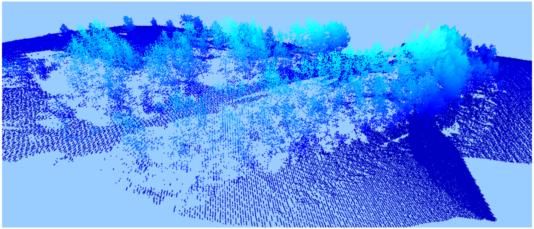

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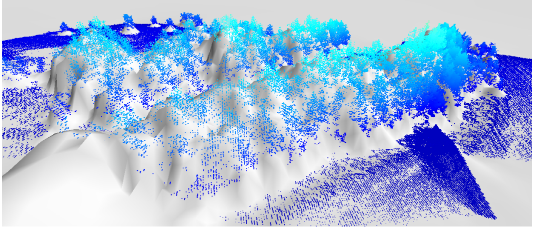

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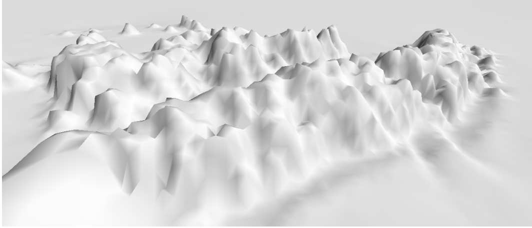
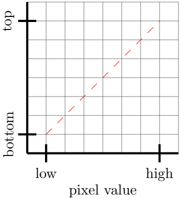


Editing and Restoration

- Intensity Adjustment
- Denoising
- Inpainting
- Deblurring
- Distortion Correction



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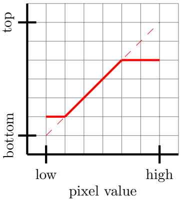
washing out colors



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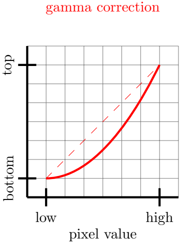

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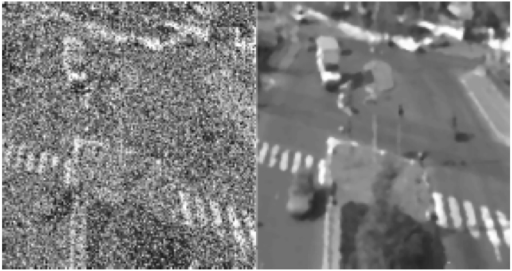

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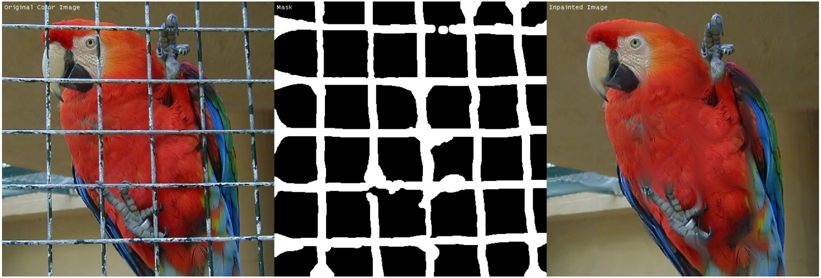

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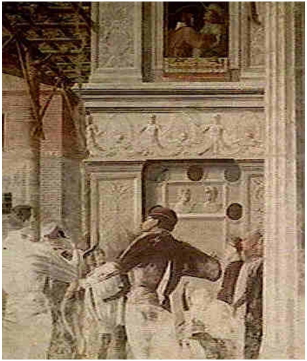

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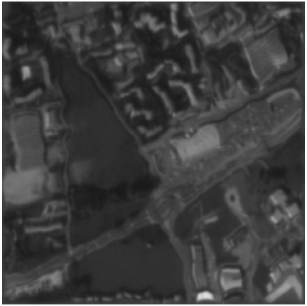
Reference image

Image to be restored



SPOT 5 HRG satellite with Hypermode sampling

Image Compression

$1,024 \times 1,024 = 1,048,576$ pixels

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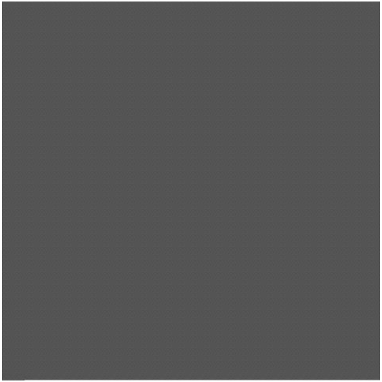

1 wavelet coefficient

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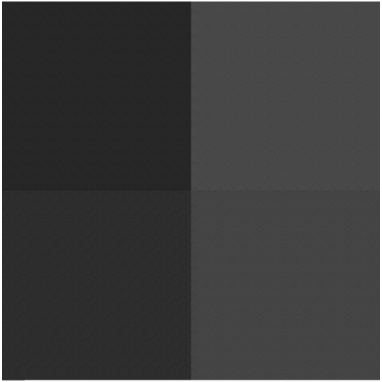

$1 + 4 = 5$ wavelet coefficients

Image Compression

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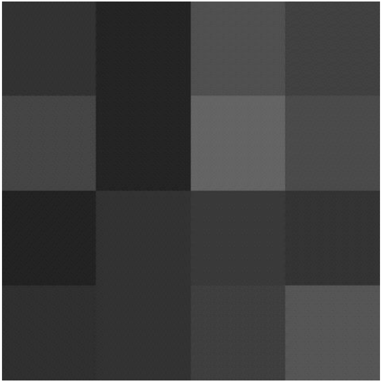

$1 + 4 + 16 = 21$ wavelet coefficients

Image Compression

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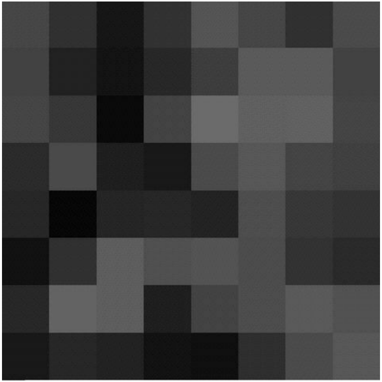

$1 + 4 + 16 + 64 = 85$ wavelet coefficients

Image Compression

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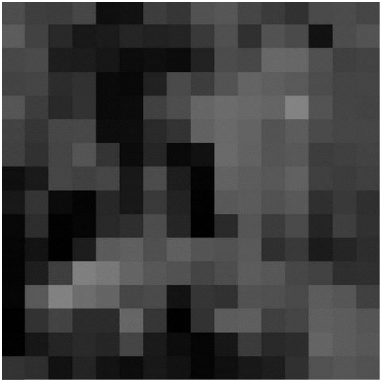

$1 + 4 + 16 + 64 + 256 = 341$ wavelet coefficients

Image Compression

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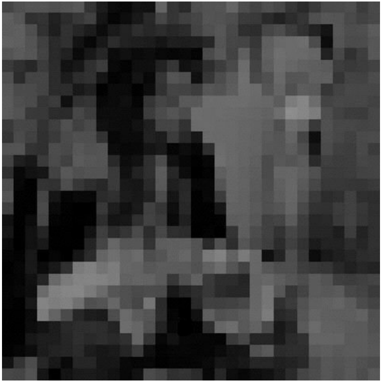

$1 + 4 + 16 + 64 + 256 + 1024 = 1365$ wavelet coefficients

Image Compression

$1,024 \times 1,024 = 1,048,576$ pixels

$1 + 4 + 16 + 64 + 256 + 1024 + 4096 = 5461$ wavelet coefficients

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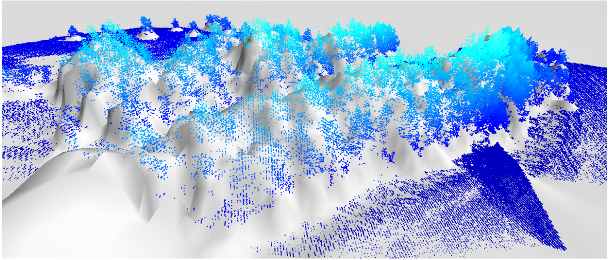
Learning Theory

Example: Data Cloud Assimilation (Dr. Binev's presentation)



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P.D.E. Models

Definition (Canny's edge points)

Smooth function $f\colon [0,1]^2 \to \mathbb{R}$.
Euclidean surface $\mathcal{S} = \big\{\big(x, y, f(x,y)\big) \in \mathbb{R}^3 : 0 \leq x, y \leq 1\big\}$.
Edges are points (x, y) with $Df(x,y)$ maximal on the gradient lines.

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P.D.E. Models

Example: Canny's Edge Detector

Definition (Canny's edge points)

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$$D^2 f\big(Df(x,y), Df(x,y)\big) = 0 \text{ and } |Df(x,y)| \text{ is large.}$$

P.D.E. Models

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Original image, followed by edge points.

P.D.E. Models

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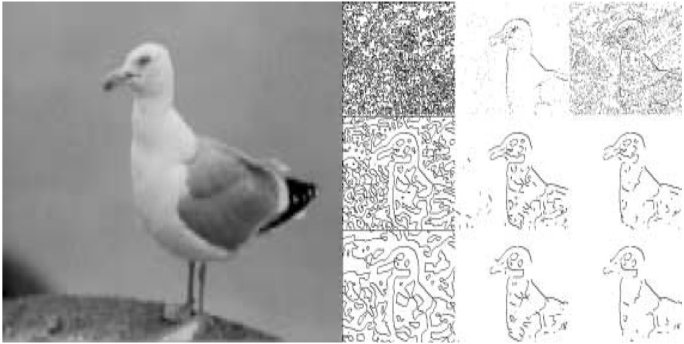
$$D^2 f\big(Df(x,y), Df(x,y)\big) = 0 \text{ and } |Df(x,y)| \text{ is large.}$$

Image smoothed out, followed by edge points.

P.D.E. Models

Example: Canny's Edge Detector

1. Convolution of the image by a Gaussian kernel
2. Estimation of $D^2 f(Df, Df)$ (finite difference scheme)
3. Convolution of $D^2 f(Df, Df)$ with a small Gauss kernel
4. Thresholding of the gradient of the result of Step 1
5. Zero-crossings of the result of the Step 3, only displayed when the threshold of Step 4 is achieved.



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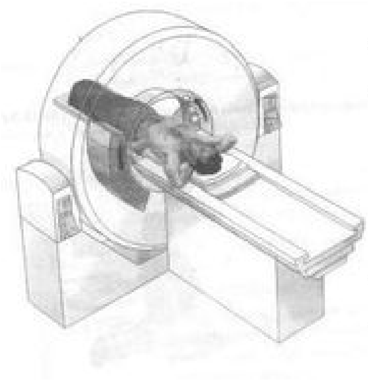
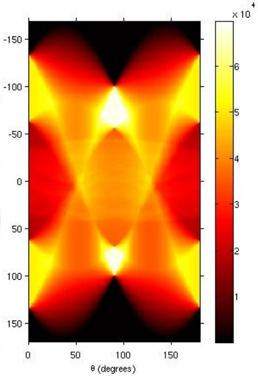
Inverse Problems

Example: X-ray Tomography





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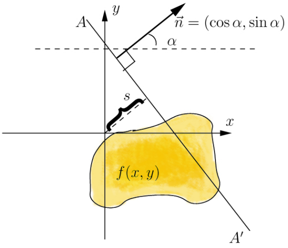


Basic Idea

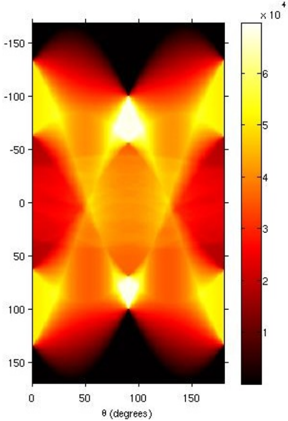
X-ray with intensity A on a line $\big(x(t), y(t)\big)$ with equation $t(\sin\alpha, -\cos\alpha) + s(\cos\alpha, \sin\alpha)$ shot through a flat object Ω with density $f(x, y)$.

Recorded intensity A' satisfies:

$$\log(A/A') = \int_{-\infty}^{\infty} f\big(x(t), y(t)\big)\, dt.$$

Inverse Problems

Example: X-ray Tomography



Basic Idea

X-ray with intensity A on a line $\big(x(t), y(t)\big)$ with equation $t(\sin\alpha, -\cos\alpha) + s(\cos\alpha, \sin\alpha)$ shot through a flat object Ω with density $f(x, y)$.
Recorded intensity A' satisfies:

$$\log(A/A') = \int_{-\infty}^{\infty} f\big(x(t), y(t)\big)\, dt.$$

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Inverse Problems

Example: X-ray Tomography

Definition (Radon Transform)

Radon transform of a function f on the plane:

$$\mathcal{R}[f](\alpha, s) = \int_{-\infty}^{\infty} f\big(x(t), y(t)\big)\, dt$$

Formal adjoint of the Radon transform:

$$\mathcal{R}^*[g](\boldsymbol{x}) = \int_0^{2\pi} g\big(\alpha, \boldsymbol{n}(\alpha) \cdot \boldsymbol{x}\big)\, d\alpha$$

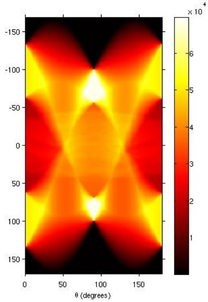
Inverse Problems

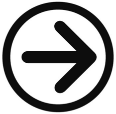
Example: X-ray Tomography

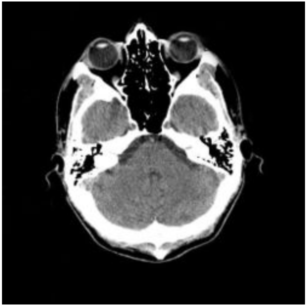
Filtered back-projection

$$f = \frac{1}{4\pi}\mathcal{R}^* H\left[\mathcal{R}f\right],$$

where $\widehat{H[h]}(\omega) = |\omega|\,\hat{h}(\omega)$ is the ramp-filter of $h\colon \mathbb{R} \to \mathbb{R}$







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Approximation Theory

Example: Computation of Smoothness

Definition (Besov Spaces)

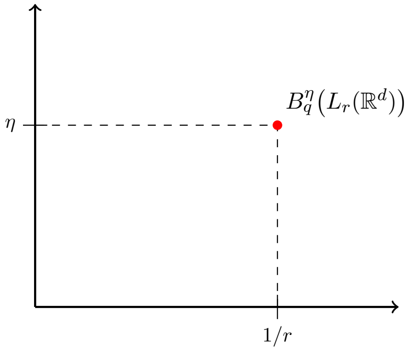
Given $f\colon \mathbb{R}^d \to \mathbb{R}$, for $h \in \mathbb{R}^d$, set for any $n \in \mathbb{N}$,

$$\Delta_h^n f(x) = \Delta_h^{n-1}\Delta_h f(x) = \sum_{k=0}^{n}(-1)^{n-k}\binom{\eta}{k} f(x+kh).$$

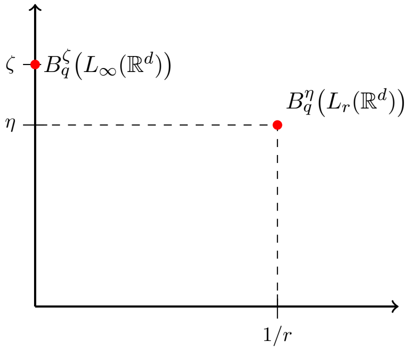
For $\eta > 0$, set $\omega_\eta(f,t)_r = \sup_{|h|<t} \left\| \Delta_h^{\lceil \eta \rceil} f \right\|_{L_r(\mathbb{R}^d)}$.
$f \in B_q^\eta\big(L_r(\mathbb{R}^d)\big)$ if

$$\|f\|_{L_r(\mathbb{R}^d)} + \left\{ \int_0^\infty \big(t^{-\eta}\omega_\eta(f,t)_r\big)^q \frac{dt}{t} \right\}^{1/q} < \infty$$

The (η, r) plane



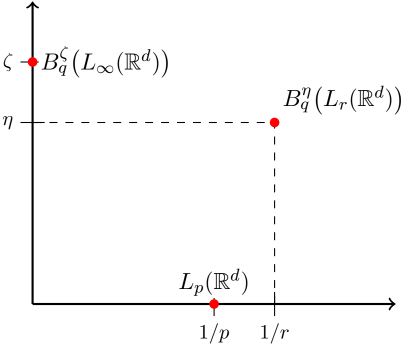
The (η, r) plane



Approximation Theory

Example: Computation of Smoothness

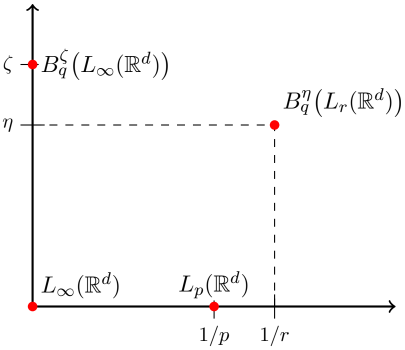
The (η, r) plane



Approximation Theory

Example: Computation of Smoothness

The (η, r) plane

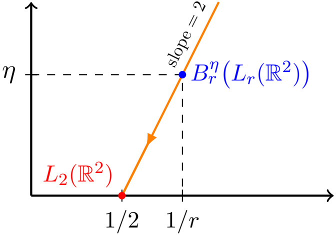


Approximation Theory

Example: Computation of Smoothness

Theorem (DeVore, Popov)

If $\eta, r > 0$ are related by $\frac{1}{r} = \frac{\eta}{2} + \frac{1}{2}$, then $B_r^\eta\big(L_r(\mathbb{R}^2)\big)$ is continuously embedded in $L_2(\mathbb{R}^2)$.



Approximation Theory

Example: Computation of Smoothness

Theorem

$f \in B_r^{\eta}\big(L_r(\mathbb{R}^2)\big)$ *if and only if* $\|f - f_N^{\mathcal{W}}\|_{L_2(\mathbb{R}^2)} = \Theta(N^{-\eta/2})$.

Approximation by selecting the N largest terms in the Wavelet decomposition

Approximation Theory

Example: Computation of Smoothness

Theorem

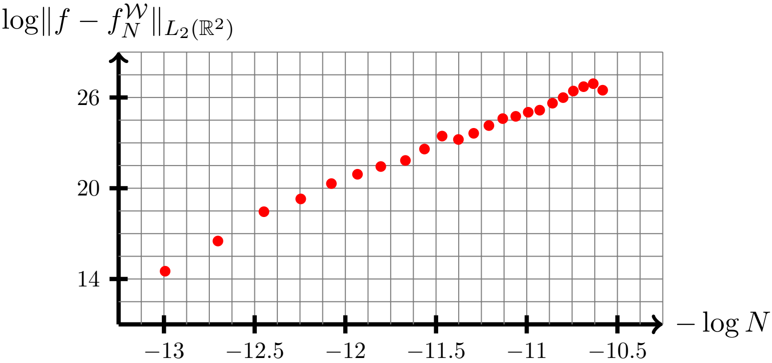
$$f \in B_r^\eta\big(L_r(\mathbb{R}^2)\big) \textit{ if and only if } \|f - f_N^{\mathcal{W}}\|_{L_2(\mathbb{R}^2)} = \Theta(N^{-\eta/2}).$$

Approximation by selecting the N largest terms in the Wavelet decomposition

or equivalently, $\log\|f - f_N^{\mathcal{W}}\|_{L_2(\mathbb{R}^2)} = \Theta\big(-\frac{\eta}{2}\log N\big).$

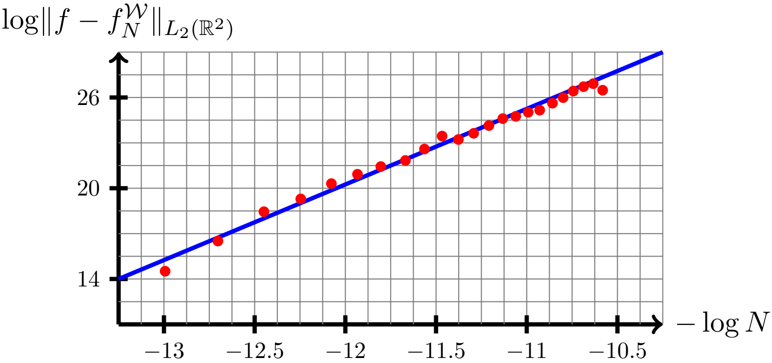
Approximation Theory

Example: Computation of Smoothness



Approximation Theory

Example: Computation of Smoothness



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Approximation Theory

Example: Computation of Smoothness

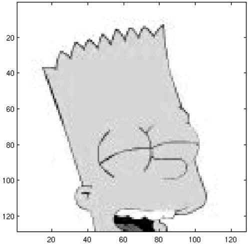
slope ≈ 0.3072

$\eta \approx 0.6144$

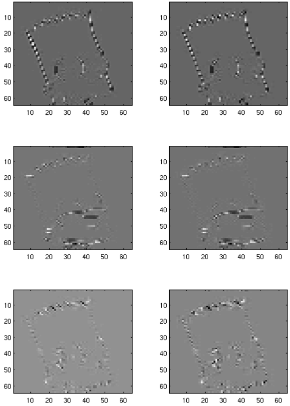
Multi-resolution Analysis

Dual-Tree Complex Wavelet Transform

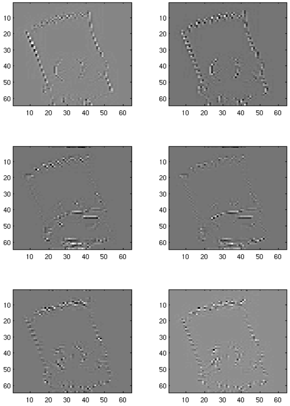
Coefficients at finest scale (6 directions)



Original Image



Real part



Imaginary part

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Multi-resolution Analysis

The Wavelet-domain hidden Markov Tree

To each wavelet coefficient $w_k = u_k + iv_k$, associate a discrete hidden state q_k that takes on values $m = \texttt{S}, \texttt{L}$ with probability mass function $p(q_k)$.

Conditioned on $q_k = m$, w_k is Gaussian with mean $\mu_{k,m}$ and variance $\sigma^2_{k,m}$. Thus, overall marginal `pdf` is

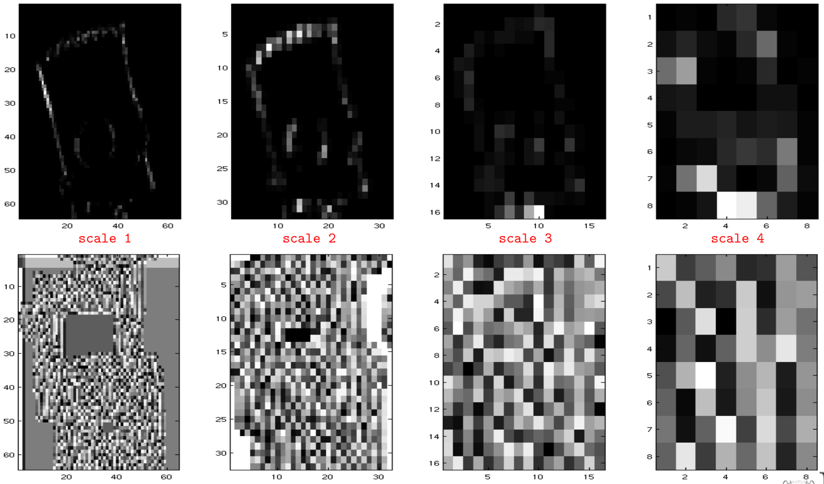
$$g(w_k) = \sum_{m=\texttt{S},\texttt{L}} p(q_k = m) g(w_k | q_k = m),$$

with $g(w_k | q_k = m) \approx \mathcal{N}(\mu_{k,m}, \sigma_{k,m})$.

Multi-resolution Analysis

Edge detection

magnitude $|w_k|$

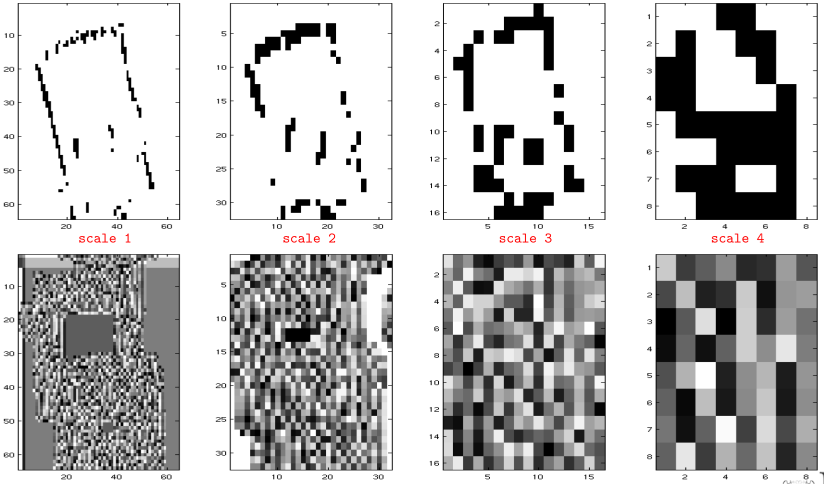


phase $\angle w_k$

Multi-resolution Analysis

Edge detection

S/L-state (S = white, L = black)

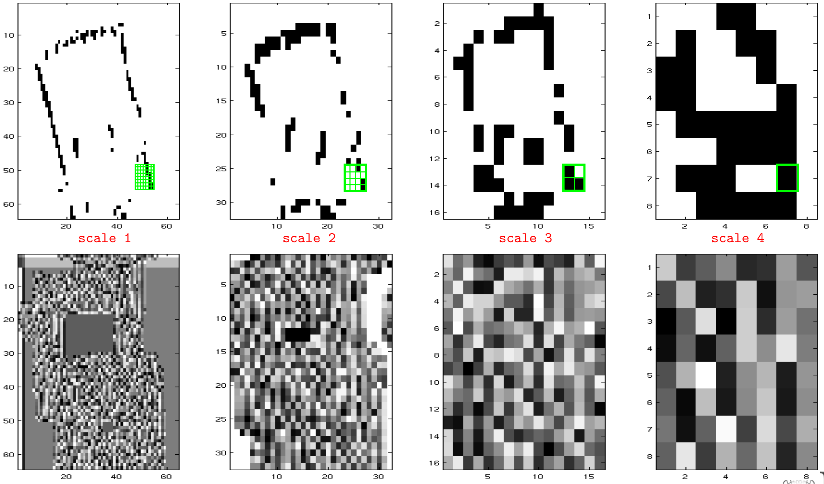


phase $\angle w_k$

Multi-resolution Analysis

Edge detection

S/L-state (S = white, L = black)

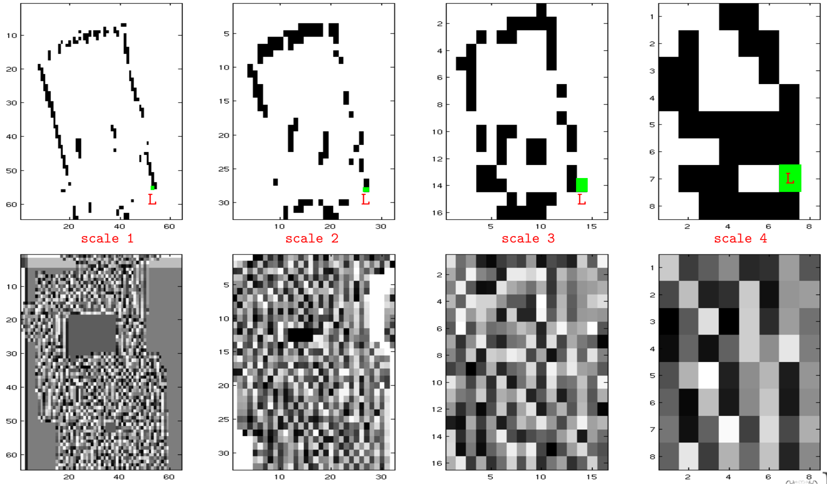


phase $\angle w_k$

Multi-resolution Analysis

Edge detection

S/L-state (S = white, L = black)

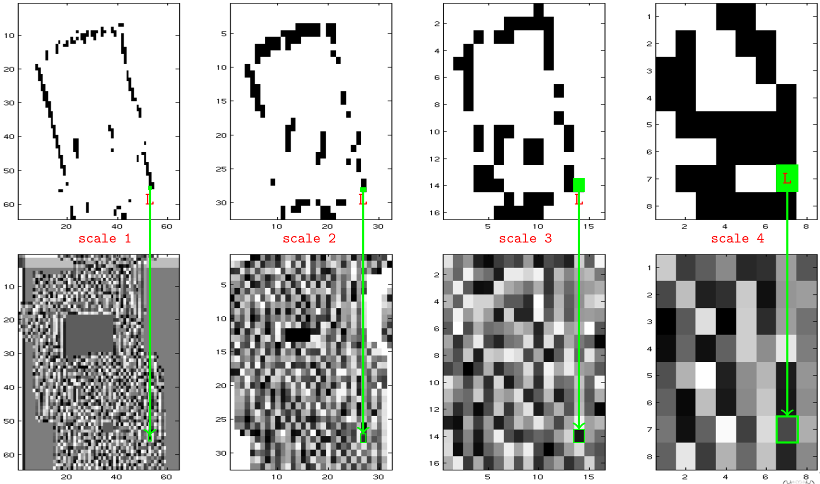


phase $\angle w_k$

Multi-resolution Analysis

Edge detection

S/L-state (S = white, L = black)



phase $\angle w_k$

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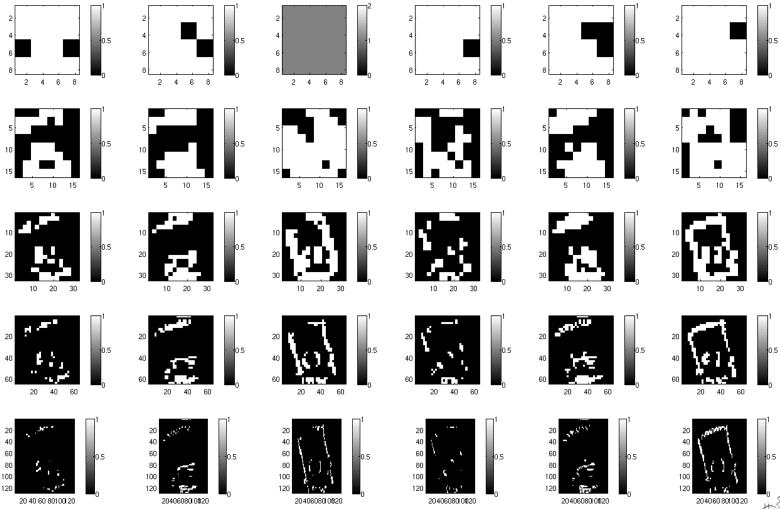
Multi-resolution Analysis

- Smooth Region. Both the coefficient and its parent have small magnitude (state `S`).
- Isolated Edge. Both coefficient and parent have large magnitude (state `L`), corresponding phases are statistically similar.
- Texture Region. Both coefficient and parent have large magnitude, but corresponding phases are very different.

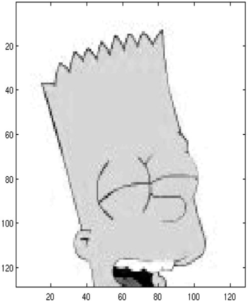
Multi-resolution Analysis

Experiments: Detection of sequences L-L-$\cdots$ with constant variance $\sigma = 0.5$

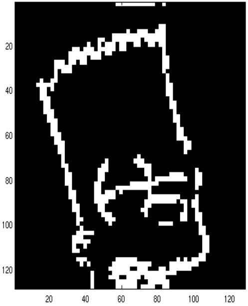


Multi-resolution Analysis

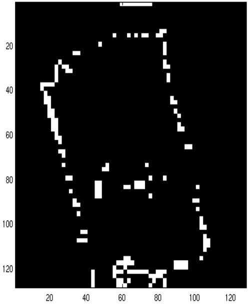
Experiments: Results depend very strongly on variance



original



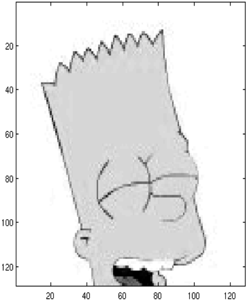
L-L-$\cdots$ map
($\sigma = 0.5$)



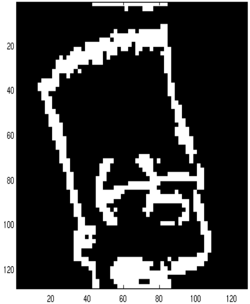
L-L-$\cdots$ map
($\sigma = 1$)

Multi-resolution Analysis

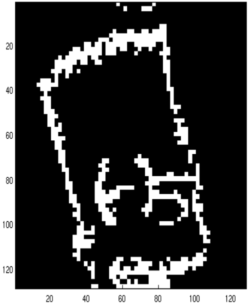
Experiments: Removal of texture region ($\sigma = 0.4$)



original



L-L-$\cdots$ map



edges
(threshold $= 2$)

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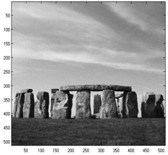
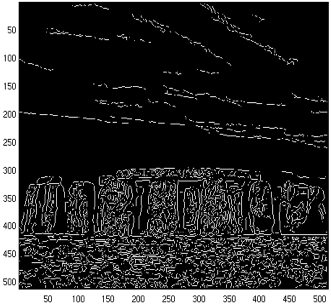
Multi-resolution Analysis

Experiments: Comparison to Canny

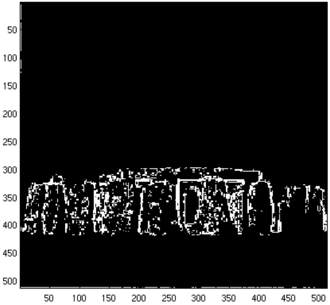


original



edge map (Canny)

`edge(IMG,'canny')`



edge map (HMM)

($\sigma = 0.3$)

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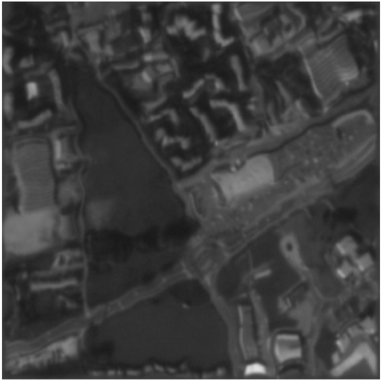
Variational Methods

Example: Restoration of Irregularly Sampled Images

Reference image

Image to be restored



SPOT 5 HRG satellite with Hypermode sampling

Variational Methods

Image Formation Model

$$g = \Gamma_\Lambda \cdot (f * h) + n$$

- f: the ideal undistorted image
- h: blurring kernel
- n: white noise with standard deviation σ
- Λ: sampling grid. $\Lambda = \mathbb{Z}^2 + \varepsilon(\mathbb{Z}^2)$
- ε: perturbation.

$$\varepsilon(x) = \sum_{k=1}^{q} a_k(x) \cos\big(2\pi\langle \xi_k, x\rangle + \phi_k\big), \quad x \in \mathbb{R}^2$$

$$\mathrm{Var}\{\varepsilon\} = A^2$$

$$\mathrm{supp}(\hat{\varepsilon}) \subseteq \big[-T_\varepsilon^{-1}, T_\varepsilon^{-1}\big] \text{ for some period } T_\varepsilon > 2$$

- Γ_Λ: sampling operator $\Gamma_\Lambda = \sum_{k\in\Lambda} \delta(\cdot - k)$.

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Variational Methods

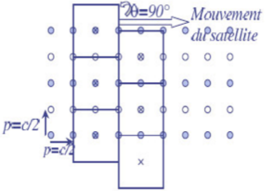
Example: Restoration of Irregularly Sampled Images

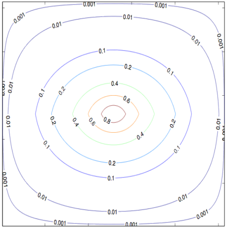
Technical details: Modulation Transfer Function

$$\widehat{h}(\boldsymbol{\xi}) = \widehat{h}_S(\boldsymbol{\xi})\widehat{h}_O(\boldsymbol{\xi})\widehat{h}_M(\boldsymbol{\xi})$$

Sensors & conductivity. Isotropic low-pass filter. Motion blur effect.

$$\widehat{h}_S(\boldsymbol{\xi}) = e^{-2\pi c \boldsymbol{\beta}\cdot\boldsymbol{\xi}} \qquad \widehat{h}_O(\boldsymbol{\xi}) = e^{-2\pi\alpha c|\boldsymbol{\xi}|} \qquad \widehat{h}_M(\boldsymbol{\xi}) = \operatorname{sinc} \tau\boldsymbol{\xi}\cdot\boldsymbol{d}$$





Variational Methods

Example: Restoration of Irregularly Sampled Images

Image Formation Model

$$g = \Gamma_\Lambda \cdot (f * h) + n$$

Solution: Variational Model

$$\min_f \left\{ \frac{1}{2} \left\| w \cdot \left(\Gamma_\Lambda \cdot (p * h * f) - g \right) \right\|^2 + \lambda \int |\nabla f| \right\}$$

- w: vector of weights used either as preconditioner or local constraints
- p: spectral projector in a low frequency region

Variational Methods

Example: Restoration of Irregularly Sampled Images

Solution: Variational Model, vector notation

$$\min_f \left\{ \frac{1}{2} \left\| W\left(SHPFf - g\right) \right\|^2 + \lambda J(f) \right\}$$

1. F: Fourier transform
2. P: Spectral projector
3. H: Convolution
4. S: Fourier coefficients to irregular samples
5. W: Spatial weights
6. J: $J(f) = \int |\nabla f|$

Variational Methods

Example: Restoration of Irregularly Sampled Images

Solution: Variational Model, Euler-Lagrange Equation

$$0 \in F^* \underbrace{P^* H^* \underbrace{S^* W^2 S}_{T} H P}_{T'} F f - F^* \underbrace{P^* H^* \underbrace{S^* W^2 g}_{\hat{r}}}_{\hat{r}'} + \lambda \partial J(f)$$

- $\partial J(f)$: subdifferential of J.

$$\partial J(f) = \left\{ x \in L^2(\Omega) : J(y) - J(f) \geq \langle x, y - f \rangle, y \in L^2(\Omega) \right\}$$

Solution: Variational Model, Evolution Equation

$$f_t = -\left[F^{-1}T'Ff - F^{-1}\hat{r}' + \lambda\partial J(f)\right]$$

Solved by implicit Euler scheme

$$f_{n+1} - f_n \in -\tau\left[F^{-1}T'Ff_{n+1} - F^{-1}\hat{r} + \lambda\partial J(f_{n+1})\right]$$