Analysis of point, line and corner singularities with the Dual-Tree Complex Wavelet Transform

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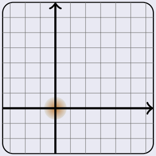
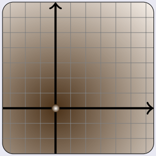
$$\langle \delta, f \rangle = f(0,0)$$



$$\langle \gamma_s, f \rangle = \int_{\mathbb{R}^2} f(\boldsymbol{x}) |\boldsymbol{x}|^{-s} \, d\boldsymbol{x}$$

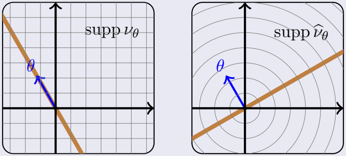
$$(0 < s < 2)$$



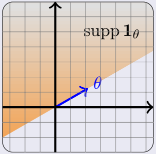
Tempered distributions modeling singularities

Line Singularities

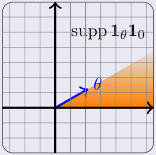
$$\langle \nu_\theta, f \rangle = \int_{\mathbb{R}} f(x\cos\theta, x\sin\theta)\, dx$$



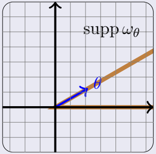
$$\langle \mathbf{1}_\theta, f \rangle = \int_{\{y > x\tan\theta\}} f(x, y)\, dy\, dx$$



$$\langle \Omega_\theta, f \rangle = \langle \mathbf{1}_\theta \mathbf{1}_0, f \rangle = \int_{\text{wedge}} f(x,y)\,dy\,dx$$



$$\langle \omega_\theta, f \rangle = \int_{\delta\{\text{wedge}\}} f(x,y)\,dy\,dx$$



Definition

- $\{\phi, \psi\}$ (scaling function, mother wavelet, 1D)
- $\{\mathcal{H}\phi, \mathcal{H}\psi\}$ does not give wavelets.
- Approx. $\mathcal{H}$ by $\boldsymbol{H}$ so that $\{\boldsymbol{H}\phi, \boldsymbol{H}\psi\}$ gives wavelets.
- Build complex valued wavelets (1D):

$$\Phi = \phi + i\boldsymbol{H}\phi, \qquad \Psi = \psi + i\boldsymbol{H}\psi.$$

- Build complex valued wavelets (2D):

$$\Phi \otimes \Psi, \quad \Psi \otimes \Phi, \quad \Psi \otimes \Psi, \quad \Phi \otimes \overline{\Psi}, \quad \Psi \otimes \overline{\Phi}, \quad \Psi \otimes \overline{\Psi}.$$

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$$
\begin{aligned}
|a|(\Psi\otimes\Psi)_{a\boldsymbol{b}} &= \Big[\psi\big(\tfrac{x-b_1}{a}\big) + i\boldsymbol{H}\psi\big(\tfrac{x-b_1}{a}\big)\Big]\Big[\psi\big(\tfrac{y-b_2}{a}\big) + i\boldsymbol{H}\psi\big(\tfrac{y-b_2}{a}\big)\Big] \\
&= \Big[\psi\big(\tfrac{x-b_1}{a}\big)\psi\big(\tfrac{y-b_2}{a}\big) - \boldsymbol{H}\psi\big(\tfrac{x-b_1}{a}\big)\boldsymbol{H}\psi\big(\tfrac{y-b_2}{a}\big)\Big] \\
&\quad + i\Big[\psi\big(\tfrac{y-b_2}{a}\big)\boldsymbol{H}\psi\big(\tfrac{x-b_1}{a}\big) + \psi\big(\tfrac{x-b_1}{a}\big)\boldsymbol{H}\psi\big(\tfrac{y-b_2}{a}\big)\Big] \\
\Re(\Psi\otimes\Psi)_{a\boldsymbol{b}} &= \frac{1}{|a|}\Big[\psi\big(\tfrac{x-b_1}{a}\big)\psi\big(\tfrac{y-b_2}{a}\big) - \boldsymbol{H}\psi\big(\tfrac{x-b_1}{a}\big)\boldsymbol{H}\psi\big(\tfrac{y-b_2}{a}\big)\Big] \\
\Im(\Psi\otimes\Psi)_{a\boldsymbol{b}} &= \frac{1}{|a|}\Big[\psi\big(\tfrac{y-b_2}{a}\big)\boldsymbol{H}\psi\big(\tfrac{x-b_1}{a}\big) + \psi\big(\tfrac{x-b_1}{a}\big)\boldsymbol{H}\psi\big(\tfrac{y-b_2}{a}\big)\Big]
\end{aligned}
$$

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Analysis away from **0**

$\boldsymbol{b} \neq \mathbf{0}$,

$$\langle \delta, \Re(\Psi \otimes \Psi)_{a\boldsymbol{b}} \rangle = \frac{1}{|a|}\Big[\psi\big(-\frac{b_1}{a}\big)\psi\big(-\frac{b_2}{a}\big) - \boldsymbol{H}\psi\big(-\frac{b_1}{a}\big)\boldsymbol{H}\psi\big(-\frac{b_2}{a}\big)\Big]$$

$$\langle \delta, \Im(\Psi \otimes \Psi)_{a\boldsymbol{b}} \rangle = \frac{1}{|a|}\Big[\psi\big(-\frac{b_1}{a}\big)\boldsymbol{H}\psi\big(-\frac{b_2}{a}\big) + \psi\big(-\frac{b_2}{a}\big)\boldsymbol{H}\psi\big(-\frac{b_1}{a}\big)\Big]$$

Analysis at $\mathbf{0}$

$$\langle \delta, \Re(\Psi \otimes \Psi)_{a\mathbf{0}} \rangle = \tfrac{1}{|a|}\big(\psi(0)^2 - \boldsymbol{H}\psi(0)^2\big)$$
$$\langle \delta, \Im(\Psi \otimes \Psi)_{a\mathbf{0}} \rangle = \tfrac{1}{|a|}\psi(0)\boldsymbol{H}\psi(0)$$

Theorem

If either $\psi(0) \neq 0$, *or* $\boldsymbol{H}\psi(0) \neq 0$,

$$\big|\langle \delta, (\Psi \otimes \Psi)_{a\mathbf{0}} \rangle\big| = \mathcal{O}\big(\tfrac{1}{|a|}\big)$$
$$\sphericalangle\langle \delta, (\Psi \otimes \Psi)_{a\mathbf{0}} \rangle = \tan^{-1}\left(\frac{\psi(0)\boldsymbol{H}\psi(0)}{\psi(0)^2 - \boldsymbol{H}\psi(0)^2}\right),$$

(*constant for all* a)

Analysis away from $\mathbf{0}$

$$\langle \gamma_s, \Re(\Psi \otimes \Psi)_{a\boldsymbol{b}} \rangle = \frac{a^{2-s}}{|a|} \left\{ \iint \frac{\psi\big(u - \frac{b_1}{a}\big)\psi\big(v - \frac{b_2}{a}\big)}{(u^2+v^2)^{s/2}}\, du\, dv - \iint \frac{\boldsymbol{H}\psi\big(u - \frac{b_1}{a}\big)\boldsymbol{H}\psi\big(v - \frac{b_2}{a}\big)}{(u^2+v^2)^{2/s}}\, du\, dv \right\}$$

$$\langle \gamma_s, \Im(\Psi \otimes \Psi)_{a\boldsymbol{b}} \rangle = \frac{a^{2-s}}{|a|} \left\{ \iint \frac{\psi\big(u - \frac{b_1}{a}\big)\boldsymbol{H}\psi\big(v - \frac{b_2}{a}\big)}{(u^2+v^2)^{s/2}}\, du\, dv + \iint \frac{\psi\big(v - \frac{b_2}{a}\big)\boldsymbol{H}\psi\big(u - \frac{b_1}{a}\big)}{(u^2+v^2)^{s/2}}\, du\, dv \right\}$$

Analysis at $\mathbf{0}$

$$\langle \gamma_s, \Re(\Psi \otimes \Psi)_{a\mathbf{0}} \rangle = \frac{a^{2-s}}{|a|} \iint \frac{\psi(u)\psi(v) - \boldsymbol{H}\psi(u)\boldsymbol{H}\psi(v)}{(u^2+v^2)^{s/2}}\, du\, dv$$

$$\langle \gamma_s, \Im(\Psi \otimes \Psi)_{a\mathbf{0}} \rangle = \frac{a^{2-s}}{|a|} \iint \frac{\psi(u)\boldsymbol{H}\psi(v) + \psi(v)\boldsymbol{H}\psi(u)}{(u^2+v^2)^{s/2}}\, du\, dv$$

Theorem

$$\big|\langle \gamma_s, (\Psi \otimes \Psi)_{a\mathbf{0}} \rangle\big| = \mathcal{O}\big(\tfrac{a^{2-s}}{|a|}\big)$$

$$\sphericalangle \langle \gamma_s, (\Psi \otimes \Psi)_{a\mathbf{0}} \rangle = C_{\psi,s}, \textit{ constant for all } a$$

Analysis at a general location

$$\langle \nu_\theta, \Re(\Psi \otimes \Psi)_{a\boldsymbol{b}} \rangle = \operatorname{sign}(a) \Big\{ \int \psi\big(\cos\theta u - \tfrac{b_1}{a}\big) \psi\big(\sin\theta u - \tfrac{b_2}{a}\big)\, du - \int \boldsymbol{H}\psi\big(\cos\theta u - \tfrac{b_1}{a}\big) \boldsymbol{H}\psi\big(\sin\theta u - \tfrac{b_2}{a}\big)\, du \Big\}$$

$$\langle \nu_\theta, \Im(\Psi \otimes \Psi)_{a\boldsymbol{b}} \rangle = \operatorname{sign}(a) \Big\{ \int \psi\big(\cos\theta u - \tfrac{b_1}{a}\big) \boldsymbol{H}\psi\big(\sin\theta u - \tfrac{b_2}{a}\big)\, du - \int \psi\big(\sin\theta u - \tfrac{b_2}{a}\big) \boldsymbol{H}\psi\big(\cos\theta u - \tfrac{b_1}{a}\big)\, du \Big\}$$

Analysis at the singularity

$$\begin{aligned}&\langle \nu_\theta, \Re(\Psi \otimes \Psi)_{a\,\lambda(\cos\theta,\sin\theta)}\rangle \\ &\qquad = \operatorname{sign}(a)\Big\{ \int \psi(u\cos\theta)\psi(u\sin\theta)\,du - \\ &\qquad\qquad \int \boldsymbol{H}\psi(u\cos\theta)\boldsymbol{H}\psi(u\sin\theta)\,du \Big\}\end{aligned}$$

$$\begin{aligned}&\langle \nu_\theta, \Im(\Psi \otimes \Psi)_{a\,\lambda(\cos\theta,\sin\theta)}\rangle \\ &\qquad = \operatorname{sign}(a)\Big\{ \int \psi(u\cos\theta)\boldsymbol{H}\psi(u\sin\theta)\,du - \\ &\qquad\qquad \int \psi(u\sin\theta)\boldsymbol{H}\psi(u\cos\theta)\,du \Big\}\end{aligned}$$

Theorem

$$\left|\langle \nu_\theta, (\Psi \otimes \Psi)_{a\,\lambda(\cos\theta,\sin\theta)}\rangle\right| = C_{\psi,\theta}, \textit{ constant for all } a$$
$$\sphericalangle\langle \nu_\theta, (\Psi \otimes \Psi)_{a\,\lambda(\cos\theta,\sin\theta)}\rangle = C_{\psi,\theta} \textit{ constant for all } a$$

Summary

Magnitude	Phase
$\left\vert\langle \delta, (\Psi\otimes\Psi)_{a\mathbf{0}}\rangle\right\vert = \mathcal{O}\big(\frac{1}{\vert a\vert}\big)$	$\sphericalangle\langle \delta, (\Psi\otimes\Psi)_{a\mathbf{0}}\rangle = \tan^{-1}\left(\frac{\psi(0)\boldsymbol{H}\psi(0)}{\psi(0)^2-\boldsymbol{H}\psi(0)^2}\right)$
$\left\vert\langle \gamma_s, (\Psi\otimes\Psi)_{a\mathbf{0}}\rangle\right\vert = \mathcal{O}\big(\frac{a^{2-s}}{\vert a\vert}\big)$	$\sphericalangle\langle \gamma_s, (\Psi\otimes\Psi)_{a\mathbf{0}}\rangle = C_{\psi,s}$
$\left\vert\langle \nu_\theta, (\Psi\otimes\Psi)_{a\,\lambda(\cos\theta,\sin\theta)}\rangle\right\vert = C_{\psi,\theta}$	$\sphericalangle\langle \nu_\theta, (\Psi\otimes\Psi)_{a\,\lambda(\cos\theta,\sin\theta)}\rangle = C_{\psi,\theta}$
$\left\vert\langle \mathbf{1}_\theta, (\Psi\otimes\Psi)_{a\,\lambda(\cos\theta,\sin\theta)}\rangle\right\vert = \mathcal{O}(\vert a\vert)$	$\sphericalangle\langle \mathbf{1}_\theta, (\Psi\otimes\Psi)_{a\,\lambda(\cos\theta,\sin\theta)}\rangle = C_{\psi,\theta}$

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