The Dual-Tree Complex Wavelet Transform

September 6, 2007

IMI
Industrial Mathematics Institute
University of South Carolina

Outline

Background material. Notation and Examples

Any square summable sequence $(c_n)_{n\in\mathbb{Z}}$ can be interpreted as the sequence of sampled values $f(n)$ of a band-limited function f with $\operatorname{supp}\widehat{f} \subset [-\pi, \pi]$.

$$f(x) = \sum_{n\in\mathbb{Z}} c_n \frac{\sin \pi(x-n)}{\pi(x-n)}, \qquad \widehat{f}(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{-in\xi}.$$

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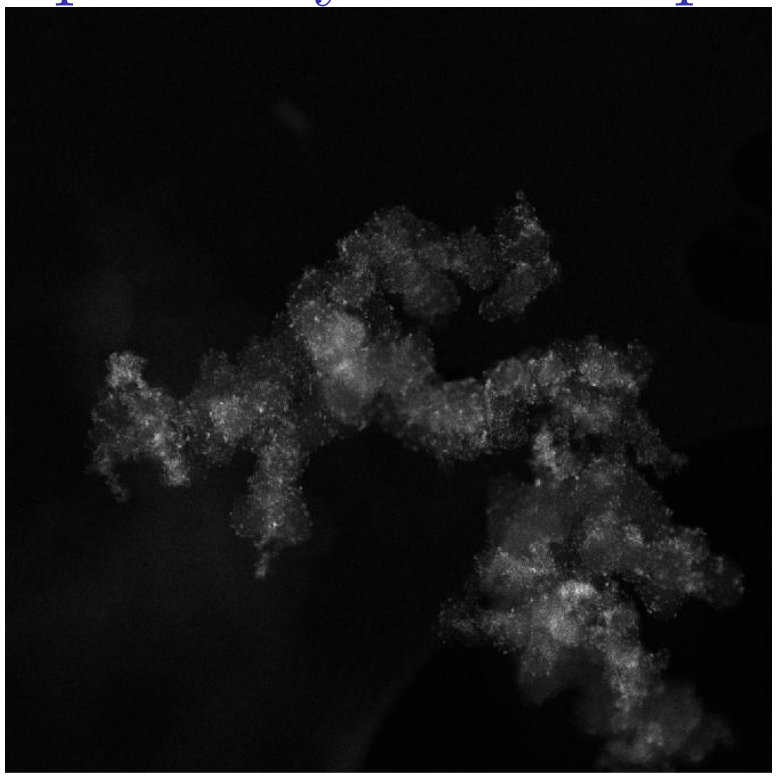
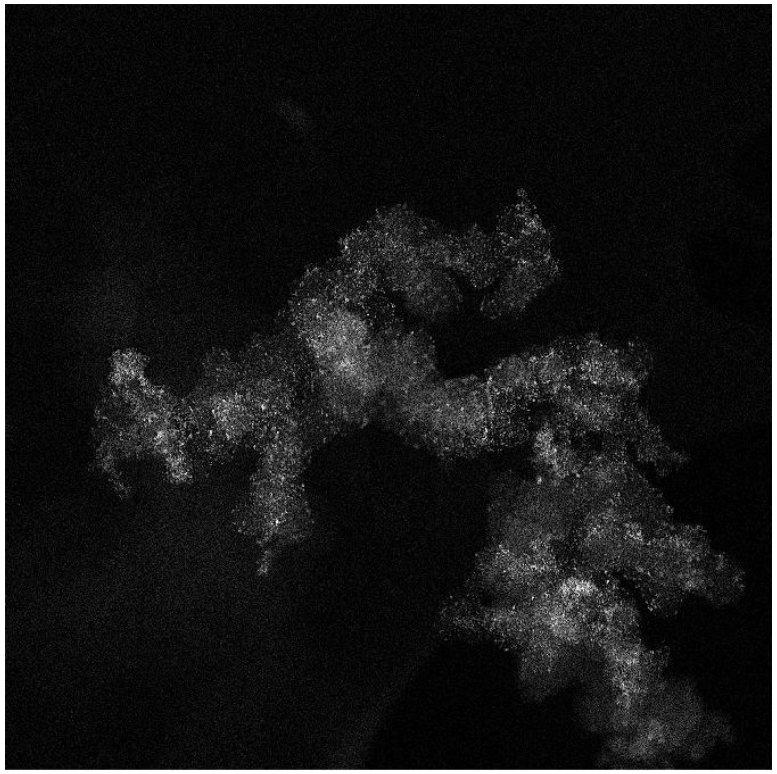
A filtering operation corresponds to the multiplication of $\widehat{f}$ with a 2π-periodic function, $\widehat{\alpha}(\xi) = \sum_{n\in\mathbb{Z}} a_n e^{-in\xi}$. The result is another band-limited function:

$$(\alpha * f)(x) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_{n-m} c_m \frac{\sin \pi(x-n)}{\pi(x-n)},$$

$$\mathcal{F}(\alpha * f)(\xi) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_{n-m} c_m e^{-in\xi}.$$

Example: 3-by-3 unsharp contrast enhancement filter





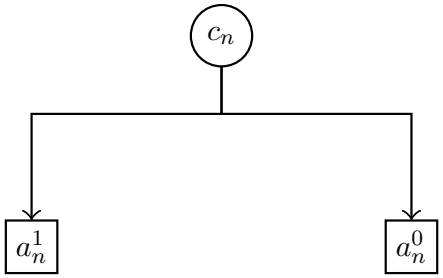
$$
\begin{tikzcd}
f(x) \arrow[rr, "\boxed{\widehat{\alpha}(\xi)}"] & & (\alpha * f)(x) \\
(c_n)_{n\in\mathbb{Z}} \arrow[u, leftrightarrow, "\text{signal}"] \arrow[rr, "\boxed{(a_n)_{n\in\mathbb{Z}}}"] & & \left(\sum_{m\in\mathbb{Z}} a_{m-n} c_m\right)_{n\in\mathbb{Z}}
\end{tikzcd}
$$

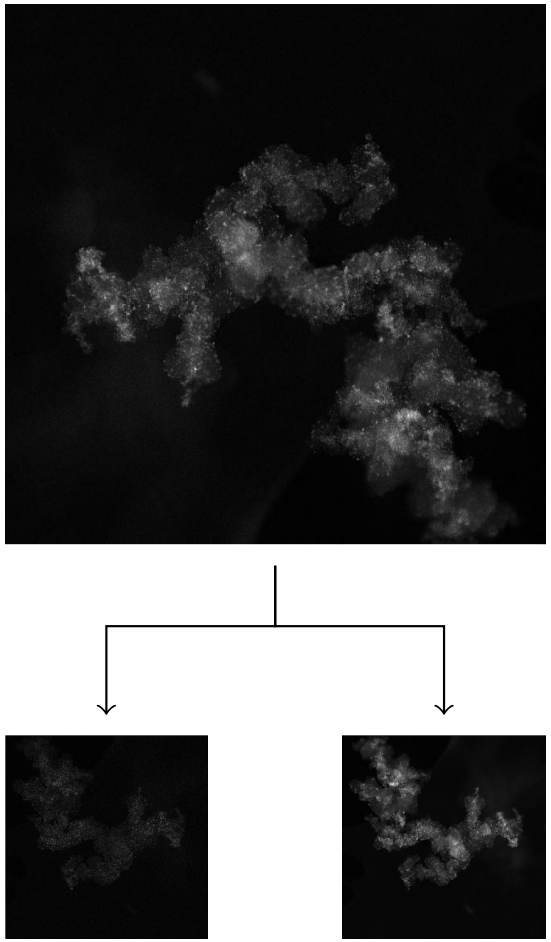
transfer function / frequency response

filter / impulse response

Subband Filtering Schemes

In signal processing an incoming signal is often decomposed into different frequency bands after which they can then be coded and transmitted separately and efficiently. This decomposition of a signal is usually done using a collection of filters called a filter bank.





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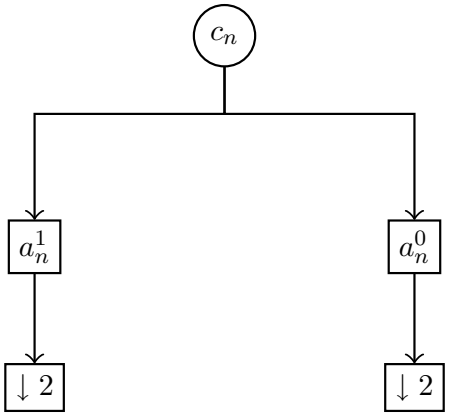
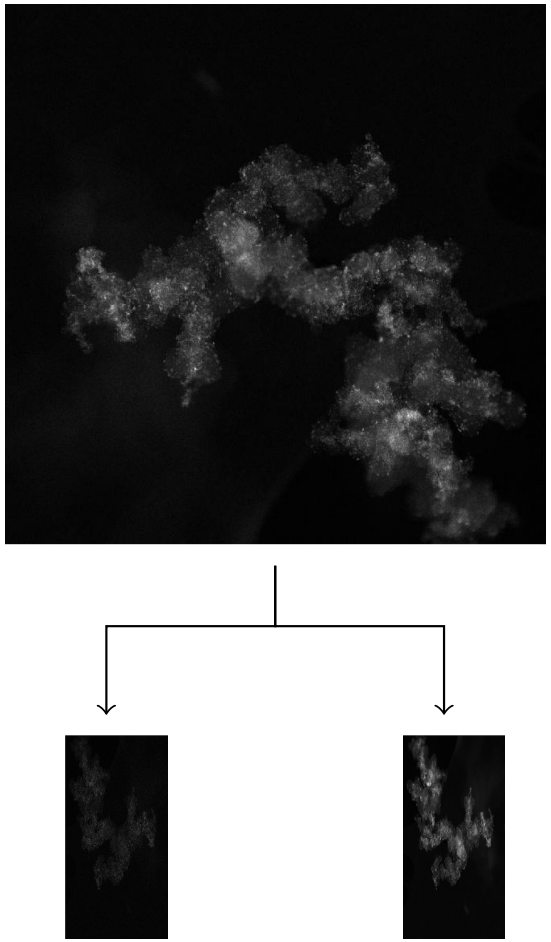
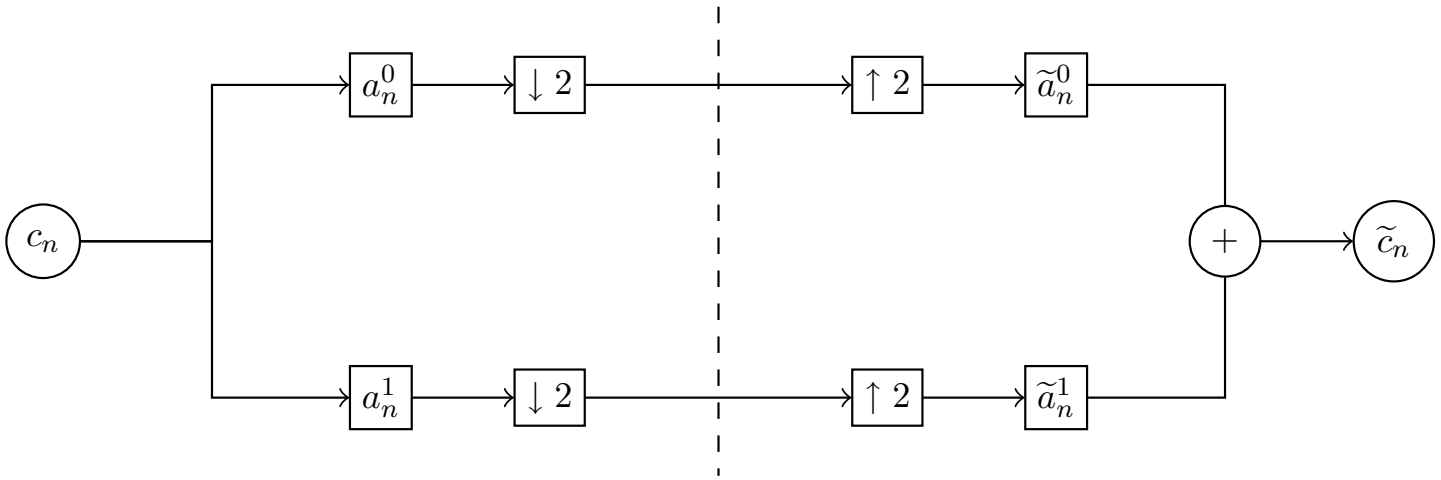




Reconstruction



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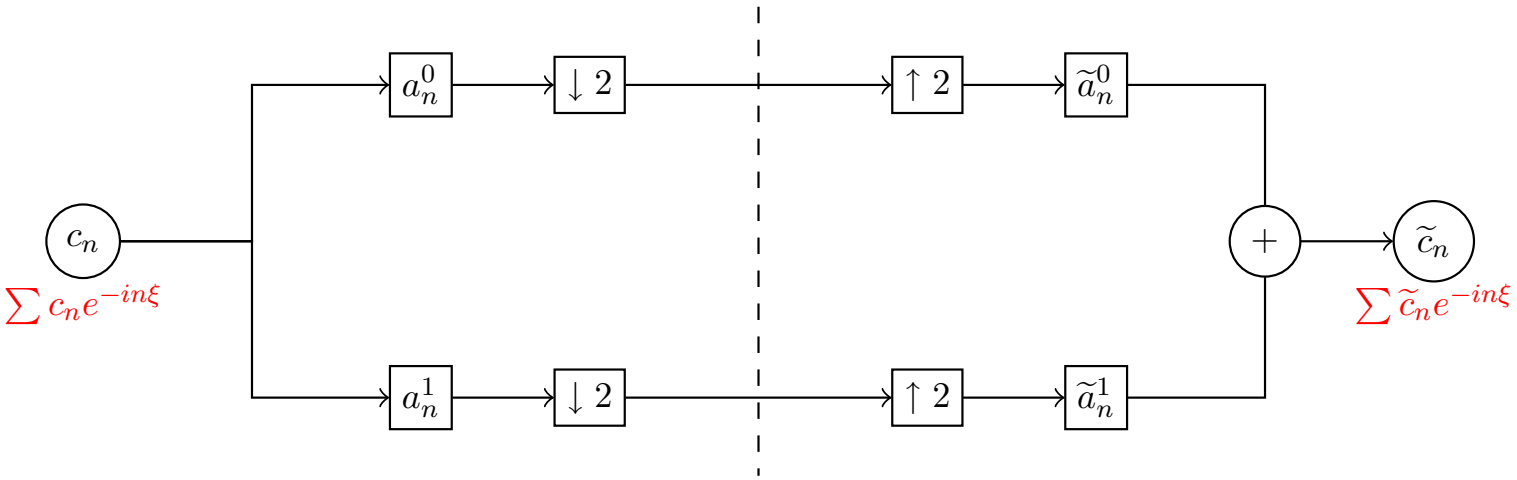
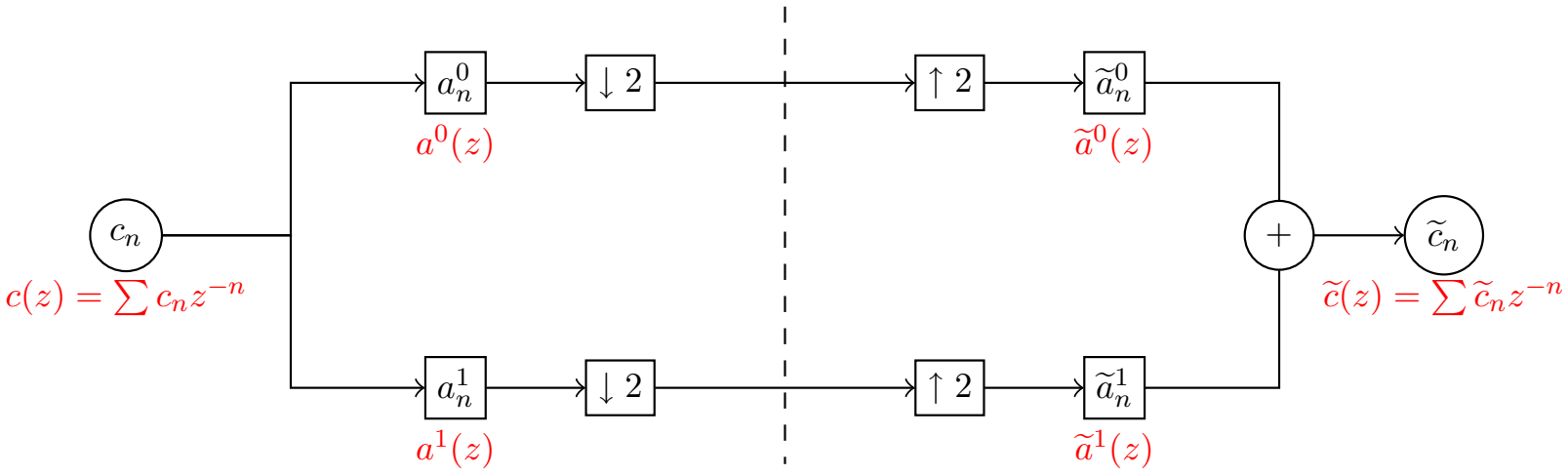


Reconstruction



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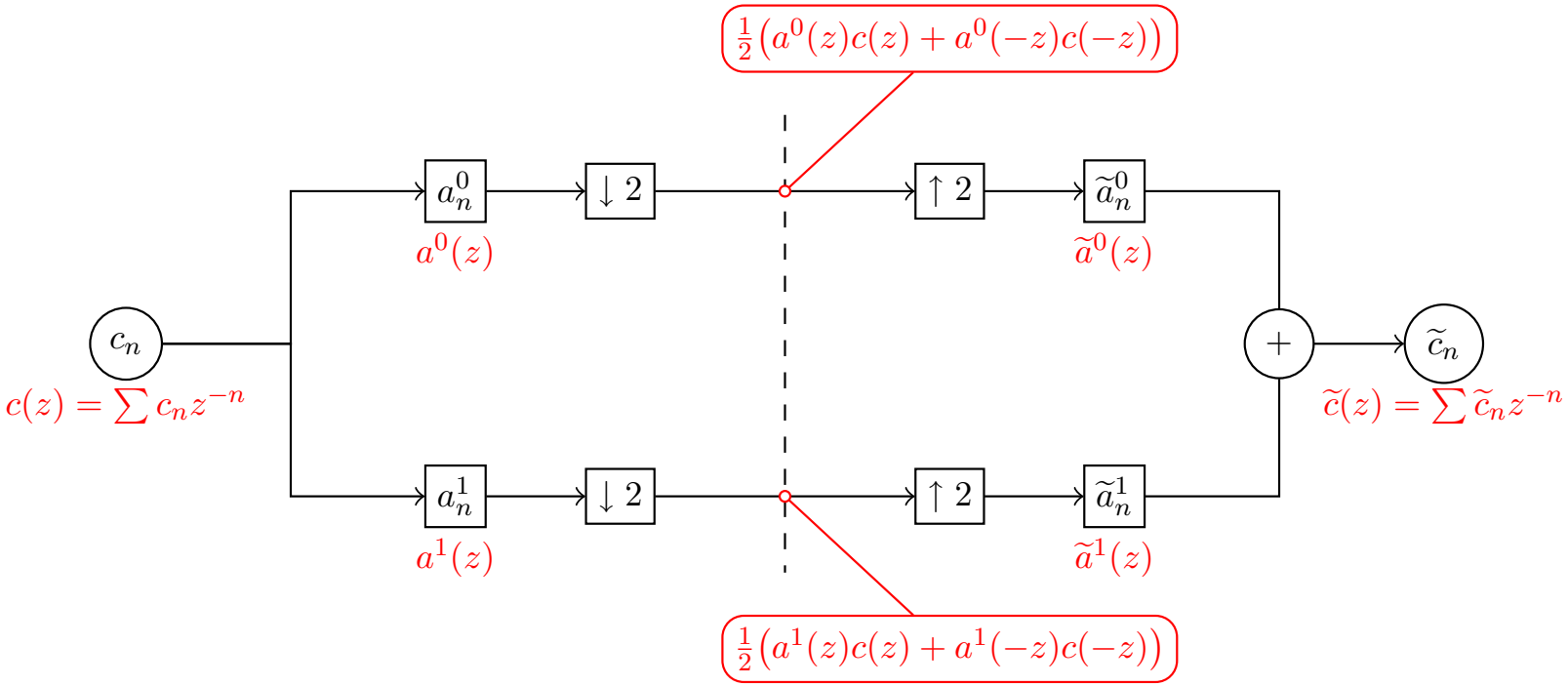
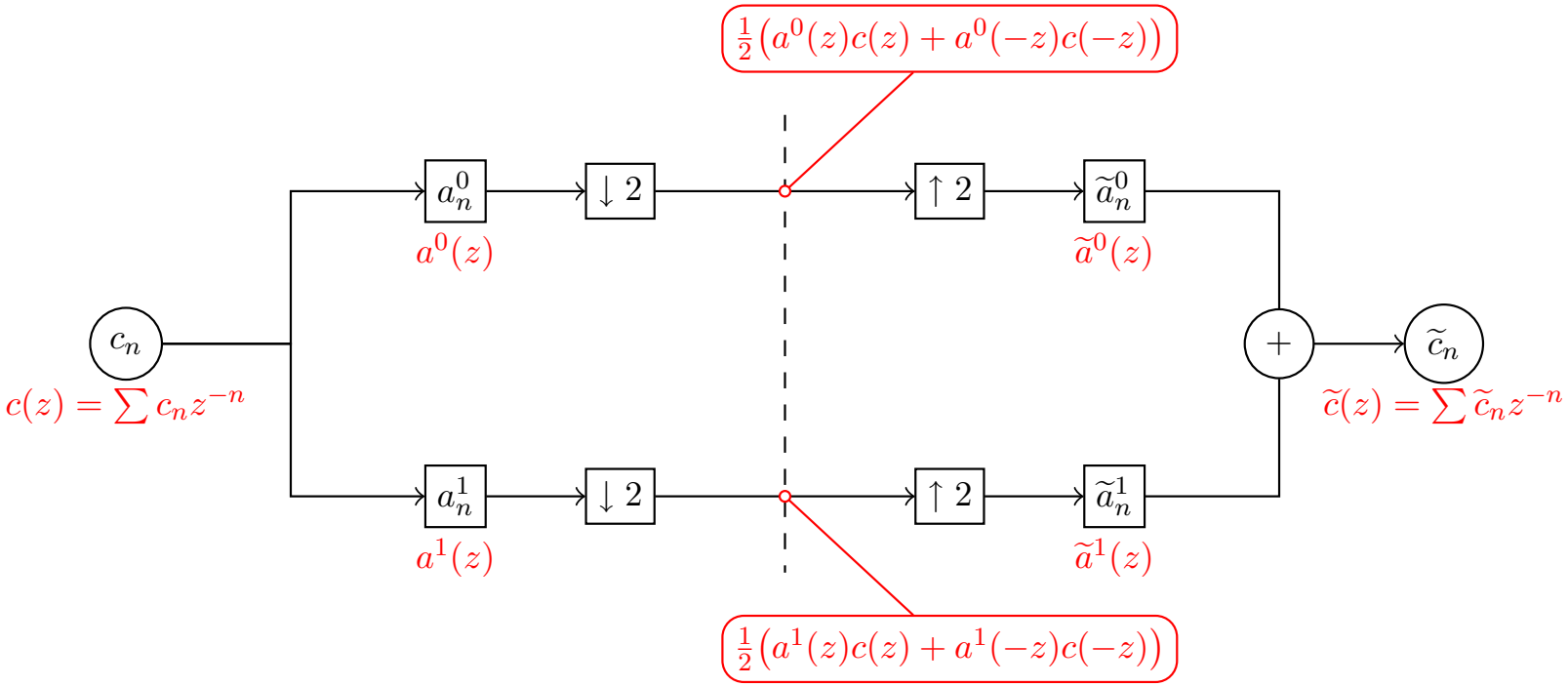
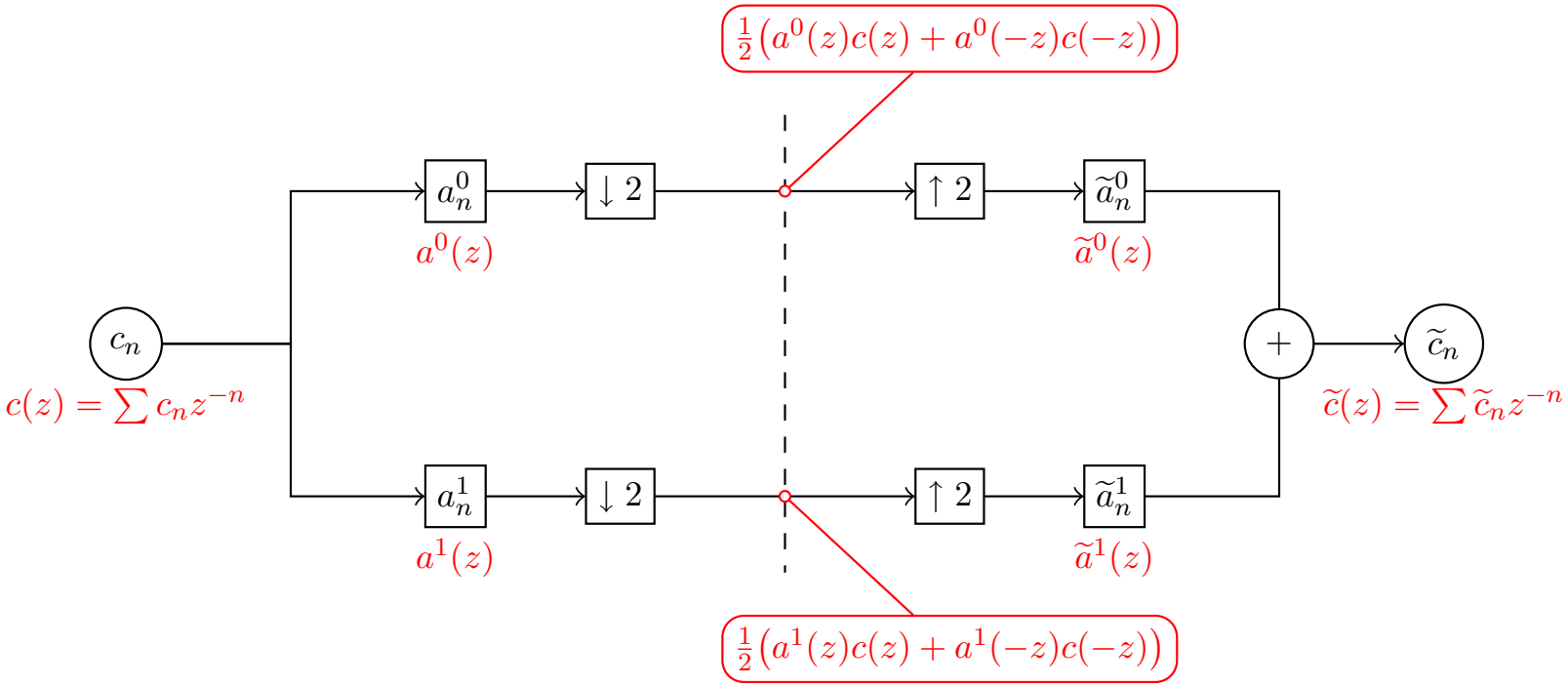


Reconstruction



$$\widetilde{c}(z) = \tfrac{1}{2}\big(\widetilde{a}^0(z)a^0(z) + \widetilde{a}^1(z)a^1(z)\big)c(z) + \underbrace{\tfrac{1}{2}\big(\widetilde{a}^0(z)a^0(-z) + \widetilde{a}^1(z)a^1(-z)\big)c(-z)}_{\text{aliasing effects}}$$

Perfect Reconstruction



$$\boxed{\widetilde{a}^0(z)a^0(z) + \widetilde{a}^1(z)a^1(z) = 2, \qquad \widetilde{a}^0(z)a^0(-z) + \widetilde{a}^1(z)a^1(-z) = 0}$$

Wavelets

$1,024 \times 1,024 = 1,048,576$ pixels

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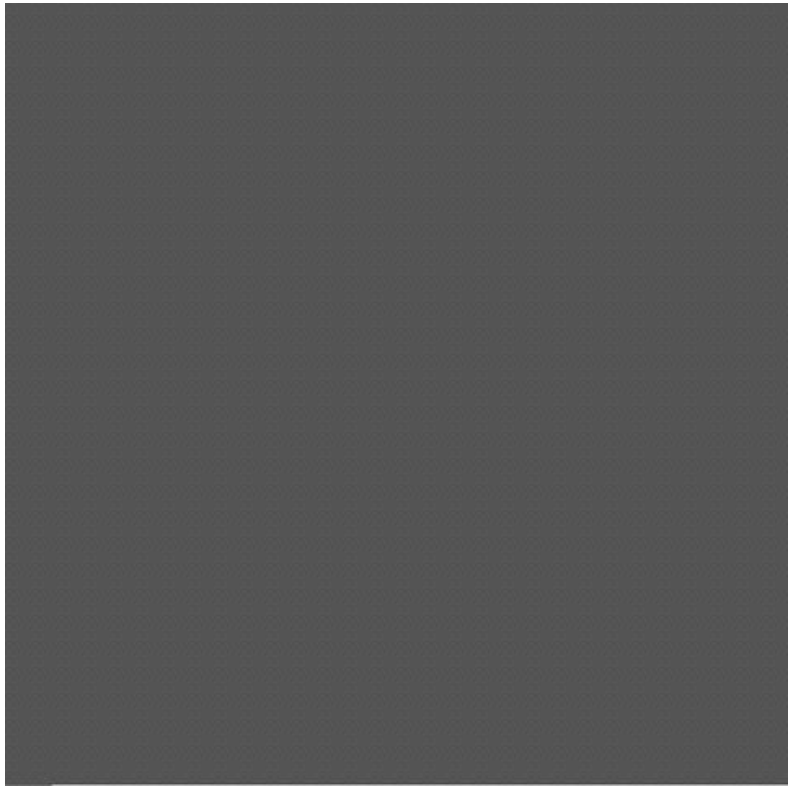

1 wavelet coefficient

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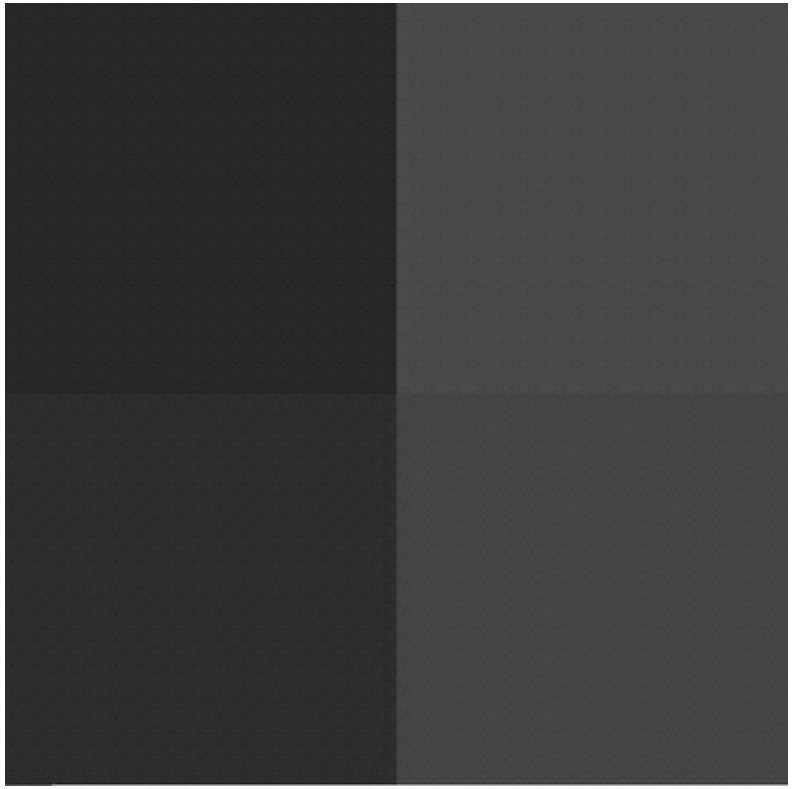

$1 + 4 = 5$ wavelet coefficients

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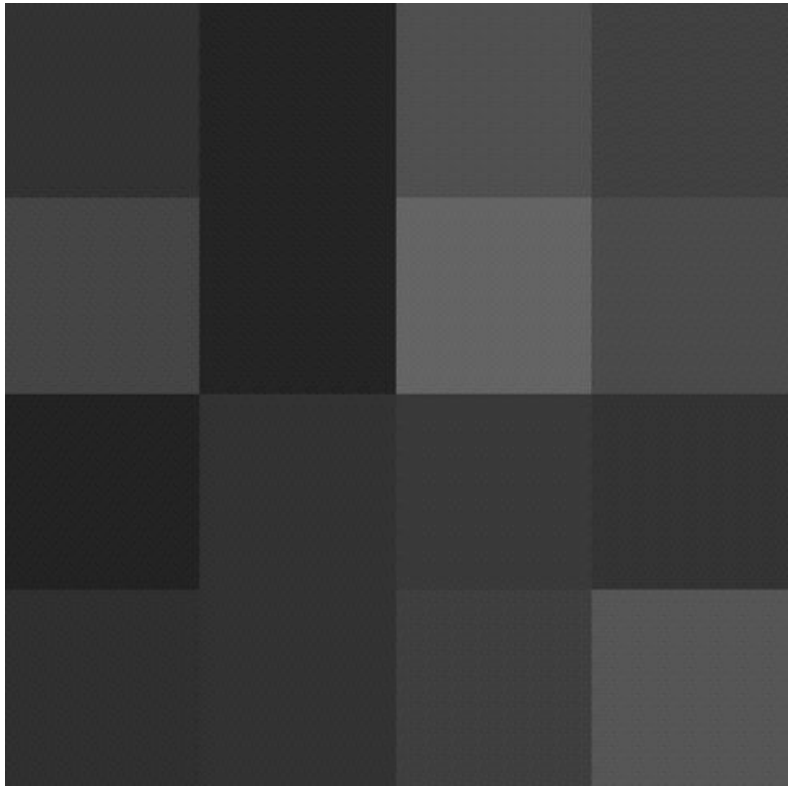

$1 + 4 + 16 = 21$ wavelet coefficients

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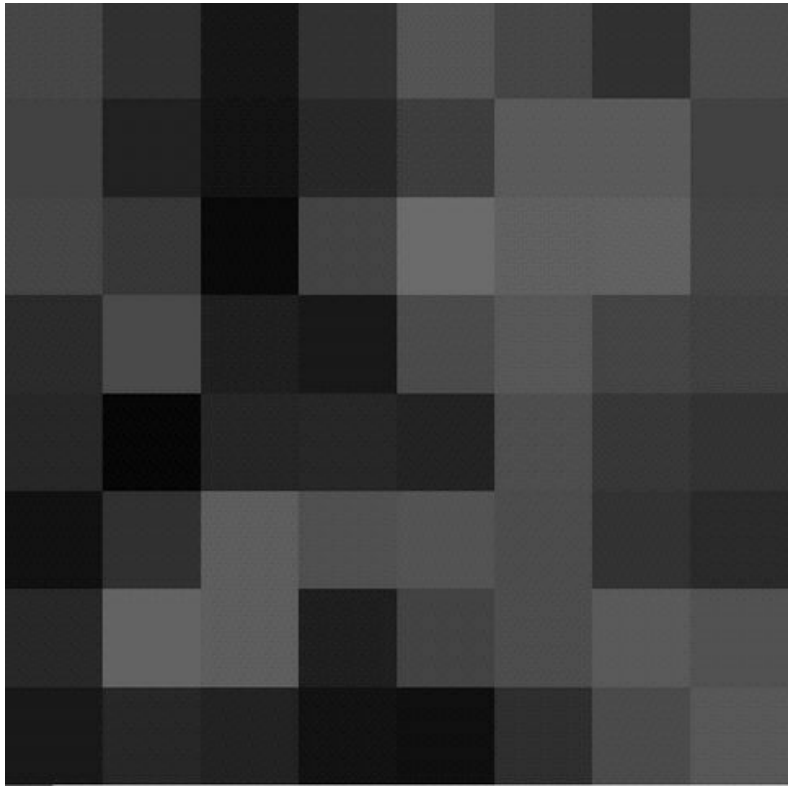

$1 + 4 + 16 + 64 = 85$ wavelet coefficients

Wavelets

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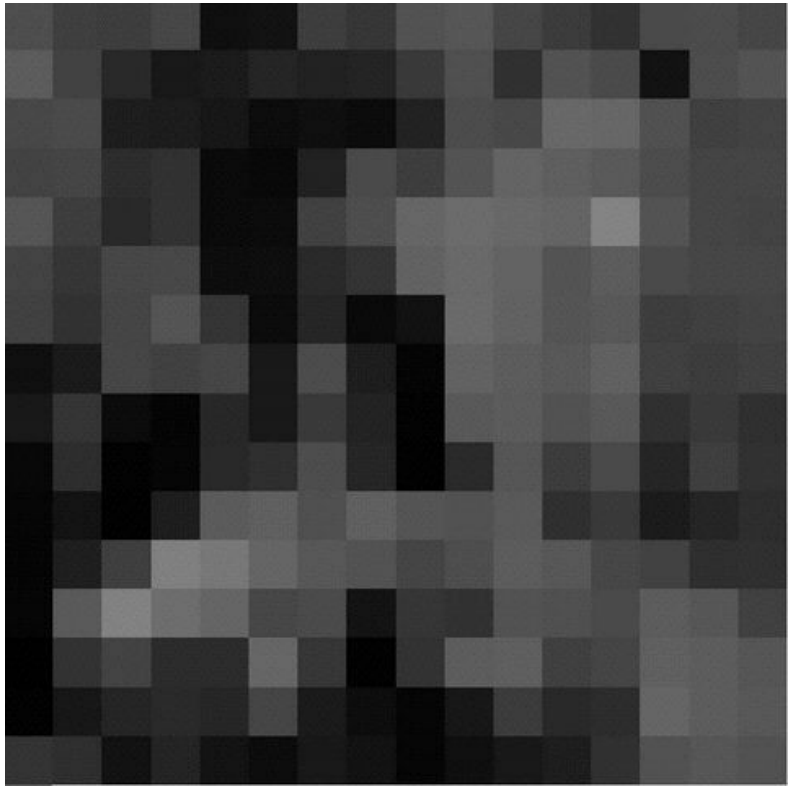


$1 + 4 + 16 + 64 + 256 = 341$ wavelet coefficients

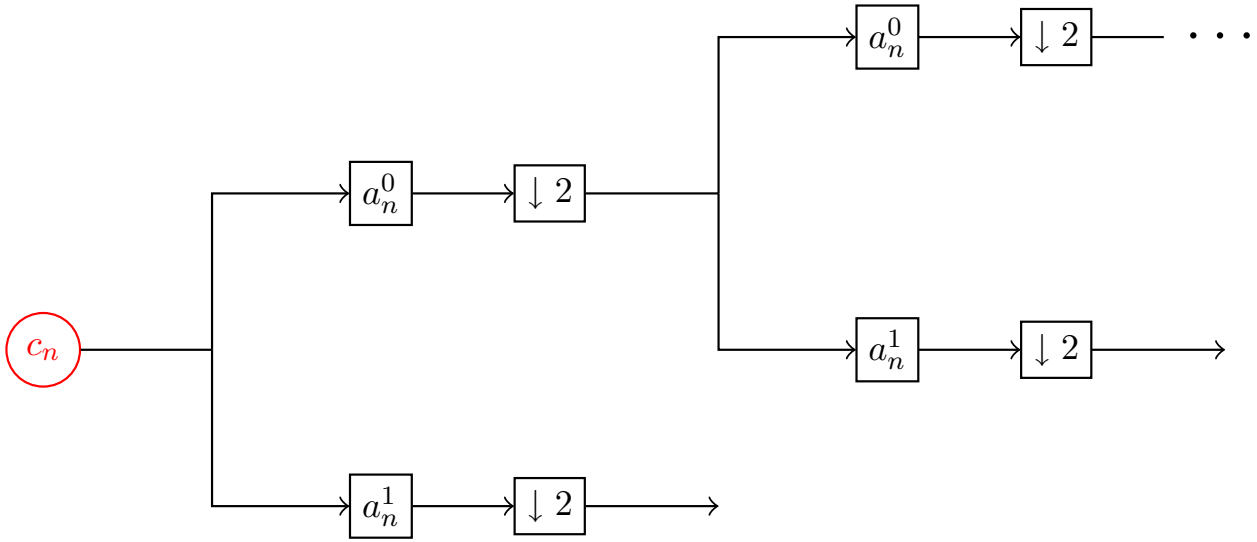
Wavelets

$1,024 \times 1,024 = 1,048,576$ pixels

$21,845$ wavelet coefficients

Connection with Subband Filtering Schemes

Multiresolution analysis leads to a hierarchical scheme for the computation of the wavelet coefficients of a function:



Problems with Real Wavelets

Shift Variance

A small shift of the signal causes major variations in the distribution of energy between wavelet coefficients at different scales.

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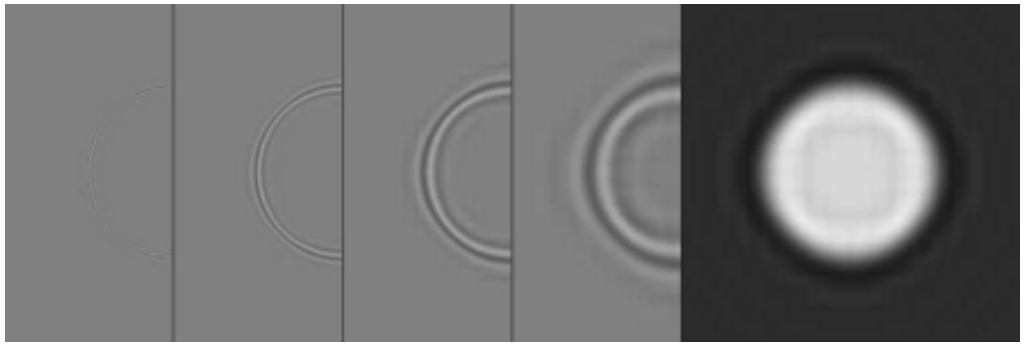
Poor Directional Selectivity

The standard tensor-product construction of multi-variate wavelets produces a checkerboard pattern that is simultaneously oriented along several directions. This lack of directional selectivity complicates processing of geometric image features like ridges and edges.

N. Kingsbury: "Complex Wavelets for Shift Invariant Analysis and Filtering of Signals"

Some Complex Wavelets do not have those problems!



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The key: "Hilbert Transform pairs"

Complex-valued scaling function $\phi\colon \mathbb{R} \to \mathbb{C}$ and complex-valued wavelet $\psi\colon \mathbb{R} \to \mathbb{C}$ satisfying

$$\psi(t) = u(t) + i\mathcal{H}u(t).$$

You get extra points if $u\colon \mathbb{R} \to \mathbb{R}$ is even, and $\mathcal{H}u$ is odd.

Watch out! Not so easy to code

For a complex-valued function $\psi(t) = u(t) + i\mathcal{H}u(t)$,

$$\begin{aligned}\widehat{\psi}(\xi) &= \widehat{u}(\xi) + i\mathcal{F}\big(\mathcal{H}u\big)(\xi) \\ &= \widehat{u}(\xi) - \operatorname{sign}(\xi)\widehat{u}(\xi) \\ &= \begin{cases} 0 & \text{if } \xi > 0, \\ \widehat{u}(0) & \text{if } \xi = 0, \\ 2\widehat{u}(\xi) & \text{if } \xi < 0. \end{cases}\end{aligned}$$

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Neither $a^0(z)$ nor $\widetilde{a}^0(x)$ is a reasonable low-pass filter.

The Dual-Tree $\mathbb{C}$WT

The idea:

Require $u(t)$ to be a real-valued wavelet such that $\mathcal{H}u(t)$ is also a wavelet, and perform two different subband filtering schemes for real and imaginary parts independently.

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- The filters are real; no complex arithmetic is required for the implementation.
- The dual-tree $\mathbb{C}$WT is not critically sampled: it is two times expansive in 1-D.
- The inverse is simple: real and imaginary parts are inverted to obtain two real signals. These two signals are then averaged.