CURVELETS VS WAVELETS
Mathematical Models of Natural Images

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Image Decompositions

Let an **image** be modeled by a **function**

$$f\colon [0,1]^2 \to [0,255].$$

Decompose that function in a **Riesz basis** (wavelets, curvelets, etc.)

$$f = \sum_{\alpha} c_\alpha \psi_\alpha$$

where α is a **generalized index**.

For **wavelets**, α specifies the **scale**, the **translation**, and whether the wavelet ψ is **horizontal**, **vertical**, or **diagonal**.

For **curvelets**, α specifies the **scale**, **orientation**, and **translation** of the curvelet.

Nonlinear Approximation

Given a Riesz basis, one can consider **parametrized subsets** of $L_2(I)$:

$\Sigma_N = \{$**linear combinations** of **no more than** N Riesz basis elements$\}$.

And, we can look at the **error of approximation** by functions in those subsets:

$$E_N(f) = \inf_{S \in \Sigma_N} \|f - S\|_{L_2(I)}.$$

We can define **approximation spaces** by **how quickly** $E_N(f)$ **tends to zero**:

$$|f|_{\mathcal{A}^\alpha_\infty} = \sup_{N>0} N^\alpha E_N(f),$$

and for $0 < q < \infty$,

$$|f|_{\mathcal{A}^\alpha_q} = \left(\sum_{N>0} [N^\alpha E_N(f)]^q \frac{1}{N} \right)^{1/q}.$$

Called **nonlinear** approximation because Σ_N is **not a linear space**:

$$S_1, S_2 \in \Sigma_N \not\Rightarrow S_1 + S_2 \in \Sigma_N.$$

Approximation Spaces and Smoothness Spaces

Meta-Theorem. *Given* **any approximation method**, *there is a* **family of smoothness spaces**, *such that a function can be* **approximated to a given rate** *using that approximation method* **if and only** *if it is* **in a smoothness space** *associated with that rate of approximation.*

And vice versa.

This meta-theorem was **formalized as a tautology** by **DeVore and Popov**.

For **some** approximation methods, the smoothness spaces are **classical**.

For example, **nonlinear approximation by wavelets** yields

$$\mathcal{A}_q^\alpha = B_q^{2\alpha}(L_q(I)) \text{ when } \frac{1}{q} = \alpha + \frac{1}{2},$$

where $B_q^\alpha(L_q(I))$ is the **Besov space** with α "derivatives" in $L_q(I)$.

What are the Approximation Spaces for Curvelets?

Nobody knows. (Unless it was done recently ...)

For **some functions**, curvelets perform **much better** than wavelets (**Donoho and Candes**).

Is the **set of functions** for which **curvelets** perform at a certain rate **strictly bigger** than the set of functions for which **wavelets** perform at the same rate? I.e., is

$$\mathcal{A}_q^\alpha(\text{wavelet}) \subsetneq \mathcal{A}_q^\alpha(\text{curvelet})?$$

Nobody knows.

If f is a **"typical natural image"** with

$$f \in \mathcal{A}_{q_1}^{\alpha_1}(\text{wavelet}) \text{ and } f \in \mathcal{A}_{q_2}^{\alpha_2}(\text{curvelet}),$$

is $\alpha_1 < \alpha_2$? I.e., are **curvelets better?**

Nobody ...

But one can experiment ...

Our Experiment

Take **24 high quality images** distributed by **Kodak** in the 90s.

For both **wavelets** and **curvelets**, write

$$f = \sum_{\alpha} c_\alpha \psi_\alpha,$$

choose a **sequence of tolerances** ϵ_1, ... Let

$$N_k = \#\{\|c_\alpha \psi_\alpha\|_{L_2(I)} \geq \epsilon_k\},$$

and

$$E_k = \Big\| f - \sum_{\|c_\alpha \psi_\alpha\|_{L_2(I)} \geq \epsilon_k} c_\alpha \psi_\alpha \Big\|_{L_2(I)}.$$

If (big, bold "if") $E_k \approx C N_k^{-\alpha}$, i.e.,

$$\log E_k \approx -\alpha \log N_k + \log C,$$

then, **approximately**,

$$f \in \mathcal{A}^\alpha_\infty \text{ and } |f|_{\mathcal{A}^\alpha_\infty} \approx C.$$

Experimental Method

2048×3072 **PhotoCD** images $\xrightarrow{\text{Photoshop}}$
2048×3072 24-bit **RGB** images $\xrightarrow{\text{GraphicConverter}}$
2048×3072 8-bit **greyscale** images $\xrightarrow{\text{crop}}$
2048×2048 8-bit greyscale images.

For **curvelets** use **CurveLab 2.0** for decomposition and reconstruction.

For **wavelets** use Lucier's **biorthogonal 2-6** programs for decomposition and reconstruction. So use **scalar quantization** instead of **thresholding**, but **same theorems apply**.

Results

	WAVELETS		CURVELETS	
Image	α	**correlation**	α	**correlation**
IMG0001	-0.334137	-0.95	-0.254534	-0.97
IMG0002	-0.248966	-0.96	-0.194984	-0.98
IMG0003	-0.288569	-0.99	-0.262188	-0.99
IMG0004	-0.301217	-0.99	-0.260947	-0.99
IMG0005	-0.432991	-0.97	-0.343005	-0.96
IMG0006	-0.334473	-0.95	-0.270084	-0.96
IMG0007	-0.329154	-0.99	-0.308024	-0.99
IMG0008	-0.337559	-0.96	-0.273487	-0.98
IMG0009	-0.282722	-0.98	-0.249841	-0.99
IMG0010	-0.262158	-0.98	-0.232546	-0.99
IMG0011	-0.321989	-0.97	-0.258699	-0.98
IMG0012	-0.308411	-0.99	-0.285750	-0.99
IMG0013	-0.307241	-0.93	-0.203144	-0.93
IMG0014	-0.356034	-0.98	-0.284708	-0.98
IMG0015	-0.346455	-0.99	-0.290807	-0.99
IMG0016	-0.268068	-0.96	-0.222829	-0.98
IMG0017	-0.274453	-0.98	-0.237508	-0.99
IMG0018	-0.273756	-0.94	-0.217761	-0.96
IMG0019	-0.322174	-0.97	-0.261727	-0.99
IMG0020	-0.320285	-0.99	-0.288679	-0.99
IMG0021	-0.309660	-0.97	-0.250962	-0.98
IMG0022	-0.284261	-0.97	-0.232663	-0.98
IMG0023	-0.309470	-0.99	-0.307909	-0.99
IMG0024	-0.363456	-0.96	-0.291762	-0.97

Discussion

If we did the computations correctly, **then**:

> **Wavelets** approximate (these) “natural images” **better** than **curvelets** do.