Quiz #16

SOLUTION

1. { 12 points } **The limit of an indeterminate form as** $x \longrightarrow 0$ **can sometimes be found by expanding the functions involved in Maclaurin series and taking the limit of the series term by term. Use this method to find the limits.**

(a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1}{x^2}\left(1 - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}\right)$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left(1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)\right) = \lim_{x\to 0} \frac{1}{x^2}\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots\right)$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} x^{2k}\right) = \lim_{x\to 0} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} x^{2k-2}$$

$$= \lim_{x\to 0} \left(\frac{1}{2} - \frac{x^2}{24} + \frac{x^4}{6!} - \frac{x^6}{8!} + \ldots\right) = \frac{1}{2}$$

(b) $\displaystyle\lim_{x\to 0} \frac{\ln\sqrt{1-x}}{\sin(3x)}$

$$\lim_{x\to 0} \frac{\ln\sqrt{1-x}}{\sin(3x)} = \lim_{x\to 0} \frac{\frac{1}{2}\ln(1-x)}{\sin(3x)}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(-x)^k}{\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(3x)^{2k+1}} = \frac{1}{2} \lim_{x\to 0} \frac{-\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \ldots}{\frac{3x}{1!} - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \ldots}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{-1 - \frac{x}{2} - \frac{x^2}{3} - \ldots}{3 - \frac{3^3x^2}{3!} + \frac{3^5x^4}{5!} - \ldots} = \frac{1}{2}\left(\frac{-1+0}{3+0}\right) = -\frac{1}{6}$$