Exam #3

SOLUTIONS

1. { 15 points }

(a) **Show that the sequence is eventually strictly monotone**

$$\left\{ \frac{1}{k-\pi} \right\}_{k=0}^{\infty}$$

Let $a_k = \dfrac{1}{k-\pi}$**.** **Then** $\dfrac{a_{k+1}}{a_k} = \dfrac{k-\pi}{k+1-\pi} < 1$**,** **if** $k-\pi > 0$**.** **Thus, the sequence is eventually strictly decreasing.**

(b) **Does the series** $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k-\pi}$ **converge? Justify your answer.**

The sequence $\{a_k\}_{k=0}^{\infty}$ **is eventually strictly decreasing.**

In addition, $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k-\pi} = 0$**.**

Therefore, the alternating series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k-\pi}$ **converges.**

2. { 10 points } **Suppose that the sequence** $\{a_k\}$ **is defined recursively by**

$$a_0 = 2, \qquad a_{k+1} = 1 - \frac{a_k}{2}$$

Assuming that the sequence converges, find its limit.

Let $\displaystyle\lim_{k\to\infty} a_k = A$**.** **Then** $\displaystyle\lim_{k\to\infty} a_{k+1} = A$ **and**

$$A = 1 - \frac{A}{2} \quad \Leftrightarrow \quad \frac{3}{2}A = 1 \quad \Leftrightarrow \quad A = \frac{2}{3}$$

So, the limit is $\frac{2}{3}$**.**

3. { 30 points: +3 *points for correct answer;* −2 *points for incorrect answer;* +1 *point for no answer; the total cannot be negative.* } **Are the following statements true or false? Check the appropriate box.**

(a) **If** $\lim_{k\to\infty} a_k = 0$, **then the series** $\sum_{k=1}^{\infty} a_k$ **converges.** ☐ true **false ✓**

(b) **If** $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 3$, **then the series** $\sum_{k=1}^{\infty} a_k$ **diverges.** **✓ true** false ☐

(c) **If** $\lim_{k\to\infty} (a_k)^{\frac{1}{k}} = 1$, **then the series** $\sum_{k=1}^{\infty} a_k$ **converges.** ☐ true **false ✓**

(d) **If** $\lim_{k\to\infty} \frac{a_k}{a_{k+1}} = 2$, **then the series** $\sum_{k=1}^{\infty} a_k$ **diverges.** ☐ true **false ✓**

(e) **If** $\sum_{k=1}^{\infty} a_k = 0$, **then the series** $\sum_{k=1}^{\infty} (-1)^k a_k$ **converges.** ☐ true **false ✓**

(f) **If** $p < 0$, **then the series** $\sum_{k=1}^{\infty} (-1)^k k^p$ **converges.** **✓ true** false ☐

(g) **If** $p > 1$, **then the series** $\sum_{k=1}^{\infty} \frac{1}{k^p}$ **converges.** **✓ true** false ☐

(h) **If** $0 < q < 1$, **then the series** $\sum_{k=0}^{\infty} q^k$ **converges.** **✓ true** false ☐

(i) **If** $\sum_{k=1}^{\infty} |a_k|$ **converges, then** $\sum_{k=1}^{\infty} a_k$ **converges.** **✓ true** false ☐

(j) **If** $0 \le a_k \le b_k$ **and** $\sum_{k=1}^{\infty} a_k$ **converges, then** $\sum_{k=1}^{\infty} b_k$ **converges.**

☐ true **false ✓**

4. { 30 points } **Use any method to determine whether the series converge. Justify your answers!**

(a) $$\sum_{k=0}^{\infty} \frac{\sqrt{k}}{2k^2 - k + 3}$$ ✓ **convergent** divergent ☐

Let $a_k = \dfrac{\sqrt{k}}{2k^2 - k + 3}$. **Since** $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ **is a convergent** p**-series with** $p = \frac{3}{2} > 1$ **and** $\displaystyle\lim_{k\to\infty} \frac{a_k}{k^{-\frac{3}{2}}} = 1$, **the series** $\displaystyle\sum_{k=0}^{\infty} \frac{\sqrt{k}}{2k^2 - k + 3}$ **is convergent.**

(b) $$\sum_{k=0}^{\infty} \left(\frac{3k-2}{2k+3}\right)^k$$ ☐ convergent **divergent** ✓

The root test gives $\displaystyle\lim_{k\to\infty} (a_k)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{3k-2}{2k+3} = \frac{3}{2} > 1$ **and therefore the series is divergent.**

(c) $$\sum_{k=0}^{\infty} \frac{(-1)^k}{\ln(k+12)}$$ ✓ **convergent** divergent ☐

This is an alternating series with $a_k = \dfrac{1}{\ln(k+12)}$. **Since** a_k **is monotone decreasing and** $\displaystyle\lim_{k\to\infty} a_k = 0$, **the series is convergent.**

5. { 15 points } **Find the radius of convergence and the interval of convergence.**

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} (2x-4)^k$$

Let $a_k = \frac{(-1)^k}{\sqrt{k}}(2x-4)^k$. **Then**

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{|2x-4|^{k+1}}{\sqrt{k+1}} \frac{\sqrt{k}}{|2x-4|^k} = |2x-4| = 2|x-2|$$

and the series is absolutely convergent for $2|x-2| < 1$ **and therefore** $|x-2| < \frac{1}{2}$, **i.e. the radius of convergence is** $\frac{1}{2}$. **To check the ends of**

the interval $\left[\frac{3}{2}, \frac{5}{2}\right]$ we consider the series for $x = \frac{3}{2}$ and $x = \frac{5}{2}$. The first one is $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ which is divergent as p-series with $p = \frac{3}{2}$. The second one is $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ which is convergent as an alternating series with $\frac{1}{\sqrt{k}}$ monotone decreasing and convergent to 0. Thus, the interval of convergence is $\left(\frac{3}{2}, \frac{5}{2}\right]$.

6. { 20 points } **Find the sum of the series by associating them with some Maclaurin series.**

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{(\ln 3)^k}{k!}$

$$\sum_{k=0}^{\infty} \frac{(\ln 3)^k}{k!} = e^{\ln 3} = 3$$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{-1}{k\ 2^k}$

$$\sum_{k=1}^{\infty} \frac{-1}{k\ 2^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(-\frac{1}{2}\right)^k = \ln\left(1 - \frac{1}{2}\right) = \ln\frac{1}{2} = -\ln 2$$

Bonus. { 15 points } **Differentiate the Maclaurin series for** $x \sin x$ **and use the result to show that**

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+2)}{(2k+1)!}\ \pi^{2k+1} = -\pi$$

Using that the Maclaurin series for $\sin x$ is absolutely convergent for all real x we receive

$$(x \sin x)' = \frac{d}{dx}\left(x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\ x^{2k+1}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\ \frac{d}{dx}\left(x^{2k+2}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2k+2)}{(2k+1)!}\ x^{2k+1}$$

On the other hand $(x \sin x)' = \sin x + x \cos x$. For $x = \pi$ it is equal to $-\pi$ and therefore this is the value of the above series at $x = \pi$.